FACTORISATION OF TWO-VARIABLE p-ADIC L-FUNCTIONS

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ABSTRACT. Let f be a modular form which is non-ordinary at p. Loeffler has recently constructed four two-variable p-adic L-functions associated to f. In the case where $a_p = 0$, he showed that, as in the one-variable case, Pollack's plus and minus splitting applies to these new objects. In this article, we show that such a splitting can be generalised to the case where $a_p \neq 0$ using Sprung's logarithmic matrix.

1. p-ADIC LOGARITHMIC MATRICES

We first review the theory of Sprung's factorisation of one-variable p-adic L-functions in [Spr12b, Spr12a], which is a generalisation of Pollack's work [Pol03].

Let $f = \sum_{n\geq 1} a_n q^n$ be a normalised eigen-newform of weight 2 and level N with nebentypus ϵ. Fix an odd[1] prime p that does not divide N and $v_p(a_p) \neq 0$. Here, v_p is the normalised p-adic valuation with $v_p(p) = 1$. Let α and β be the two roots to

$$X^2 - a_p X + \epsilon(p)p = 0$$

with $r = v_p(\alpha)$ and $s = v_p(\beta)$. Note in particular that $0 < r, s < 1$.

Let G be a one-dimensional p-adic Lie group, which is of the form $\Delta \times \langle\gamma_p\rangle$, where Δ is a finite abelian group and $\langle\gamma_p\rangle \cong \mathbb{Z}_p$. If H is a subset of G, we write 1_H for the indicator function of H on G. Let F be a finite extension of $\mathbb{Q}_p$ that contains $\mu_{|\Delta|}$, a_n and $\epsilon(n)$ for all $n \geq 1$. For a real number $u \geq 0$, we define $D^{(u)}(G, F)$ to be the set of distributions μ on G such that for a fixed integer $n \geq 0$,

$$\inf_{g\in G} v_p\left(\mu\left(1_{g\langle\gamma_p\rangle^{p^n}}\right)\right) \geq R - un$$

for some constant $R \in \mathbb{R}$ that only depends on μ. Note that we may identify $D^{(u)}(G, F)$ with the set of power series

$$\sum_{n\geq 0}\sum_{\sigma\in\Delta} c_{\sigma,n}\sigma(\gamma_p - 1)^n$$

where $c_{\sigma,n} \in F$ and $\sup_{n>0} \frac{|c_{\sigma,n}|_p}{n^u} < \infty$ for all $\sigma \in \Delta$ (here $|\cdot|_p$ denotes the p-adic norm with $|p|_p = p^{-1}$). Let $X = \gamma_p - 1$. If η is a character on Δ, we write $e_\eta\mu$ for the η-isotypical component of μ, namely, the power series

$$\sum_{n\geq 0}\sum_{\sigma\in\Delta} c_{\sigma,n}\eta(\sigma)(\gamma_p - 1)^n \in F[[X]].$$

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[1] Our results in fact hold for $p = 2$. Since the interpolation formulae of p-adic L-functions are slightly different from the other cases, we assume $p \neq 2$ for notational simplicity.

For $\mu_1 \in D^{(u)}(\langle\gamma_p\rangle, F)$ and $\mu_2 \in D^{(u)}(G, F)$, we say that μ_1 divides μ_2 over $D^{(u)}(G, F)$ if μ_1 divides all isotypical components of μ_2 as elements in $F[[X]]$.

Definition 1.1. *We say that $(\mu_\alpha, \mu_\beta) \in D^{(r)}(G, F) \oplus D^{(s)}(G, F)$ is a pair of interpolating functions for f if for all non-trivial characters ω on G that send γ_p to a primitive p^{n-1}-st root of unity for some $n \geq 1$, there exists a constant $C_\omega \in \overline{F}$ such that*

$$\mu_\alpha(\omega) = \alpha^{-n} C_\omega \quad \text{and} \quad \mu_\beta(\omega) = \beta^{-n} C_\omega.$$

Remark 1.2. *The p-adic L-functions L_α, L_β of Amice-Vélu* [AV75] *and Višik* [Viš76] *associated to f satisfy the property stated above, with G being the Galois group* $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ *and C_ω being the algebraic part of the complex L-value $L(f, \omega^{-1}, 1)$ multiplied by some fudge factor.*

Definition 1.3. *A matrix $M_p = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix}$ with $m_{1,1}, m_{2,1} \in D^{(r)}(\langle\gamma_p\rangle, F)$ and $m_{1,2}, m_{2,2} \in D^{(s)}(\langle\gamma_p\rangle, F)$, is called a p-adic logarithmic matrix associated to f if $\det(M_p)$ is, up to a constant in $F^\times$, equal to $\log_p(\gamma_p)/(\gamma_p - 1)$, and $\det(M_p)$ divides both*

$$m_{2,2}\mu_\alpha - m_{2,1}\mu_\beta \quad \text{and} \quad -m_{1,2}\mu_\alpha + m_{1,1}\mu_\beta$$

over $D^{(1)}(G, F)$ for all interpolating functions μ_α, μ_β for f.

Lemma 1.4. *Let μ_α, μ_β be a pair of interpolating functions for f. If M_p is a p-adic logarithmic matrix associated to f, then there exist $\mu_\#, \mu_\flat \in D^{(0)}(G, F)$ such that*

$$\begin{pmatrix} \mu_\alpha & \mu_\beta \end{pmatrix} = \begin{pmatrix} \mu_\# & \mu_\flat \end{pmatrix} M_p.$$

Proof. Let

$$\mu_\# := \frac{m_{2,2}\mu_\alpha - m_{2,1}\mu_\beta}{\det(M_p)} \quad \text{and} \quad \mu_\flat := \frac{-m_{1,2}\mu_\alpha + m_{1,1}\mu_\beta}{\det(M_p)}.$$

By definition, the numerators lie inside $D^{(1)}(G, F)$ and the coefficients of $\det(M_p)$ have the same growth rate as those of $\log_p(\gamma_p)$, so $\mu_\#$ and $\mu_\flat$ lie inside $D^{(0)}(G, F)$. The factorisation follows from the fact that

$$\begin{pmatrix} m_{2,2} & -m_{1,2} \\ -m_{2,1} & m_{1,1} \end{pmatrix} M_p = \begin{pmatrix} \det(M_p) & 0 \\ 0 & \det(M_p) \end{pmatrix}.$$

□

We now recall the construction of Sprung's canonical p-adic logarithmic matrix associated to f.

Let $C_n = \begin{pmatrix} a_p & 1 \\ -\epsilon(p)\Phi_{p^n}(\gamma_p) & 0 \end{pmatrix}$, where Φ_{p^n} denotes the p^n-th cyclotomic polynomial for $n \geq 1$, $C = \begin{pmatrix} a_p & 1 \\ -\epsilon(p)p & 0 \end{pmatrix}$ and $A = \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix}$. Define

$$M_p^{(n)} := C_1 \cdots C_n C^{-n-2} A.$$

Theorem 1.5 (Sprung)**.** *The entries of the sequence of matrices $M_p^{(n)}$ converge (under the standard sup-norm on p-adic power series) in $D^{(1)}(\langle\gamma_p\rangle, F)$ as $n \to \infty$ and the limit $\lim_{n\to\infty} M_p^{(n)}$ is a p-adic logarithmic matrix associated to f.*

Proof. We only sketch our proof here since this is merely a slight generalisation of Sprung's results in [Spr12a, Spr12b].

Since $C_{n+1} \equiv C \mod (X+1)^{p^n} - 1$, we have

$$M_p^{(n+1)} \equiv M_p^{(n)} \mod (X+1)^{p^n} - 1.$$

Note that

$$A^{-1}CA = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

which implies that

$$C^{-n-2}A = A\begin{pmatrix} \alpha^{-n-2} & 0 \\ 0 & \beta^{-n-2} \end{pmatrix} = \begin{pmatrix} -\alpha^{-n-2} & -\beta^{-n-2} \\ \beta\alpha^{-n-2} & \alpha\beta^{-n-2} \end{pmatrix}. \tag{1}$$

Since all the entries in $C_1 \cdots C_n$ are integral, the coefficients of the first (respectively second) row of $M_p^{(n)}$ grow like $O(p^{-rn})$ (respectively $O(p^{-sn})$) as $n \to \infty$. Therefore, by [PR94, §1.2.1], the entries of the first (respectively second) row of $M_p^{(n)}$ converge to elements in $D^{(r)}(\langle\gamma_p\rangle, F)$ (respectively $D^{(s)}(\langle\gamma_p\rangle, F)$).

Let $M_p = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix}$ be the limit $\lim_{n\to\infty} M_p^{(n)}$. If ω is a character that sends γ_p to a primitive p^{n-1}-st root of unity, then

$$M_p(\omega) = C_1(\omega)\cdots C_{n-1}(\omega)C^{-n-1}A.$$

Note that $C_{n-1}(\omega) = \begin{pmatrix} a_p & 1 \\ 0 & 0 \end{pmatrix}$, so from (1), we see that there exist two constants $A_\omega, B_\omega \in \overline{F}$ such that

$$M_p(\omega) = \begin{pmatrix} a_pA_\omega & A_\omega \\ a_pB_\omega & B_\omega \end{pmatrix}\begin{pmatrix} -\alpha^{-n-1} & -\beta^{-n-1} \\ \beta\alpha^{-n-1} & \alpha\beta^{-n-1} \end{pmatrix} = \begin{pmatrix} -\alpha^{-n}A_\omega & -\beta^{-n}A_\omega \\ -\alpha^{-n}B_\omega & -\beta^{-n}B_\omega \end{pmatrix}.$$

In particular, if μ_α, μ_β is a pair of interpolating functions for f,

$$m_{2,2}(\omega)\mu_\alpha(\omega) - m_{2,1}(\omega)\mu_\beta(\omega) = -m_{1,2}(\omega)\mu_\alpha(\omega) + m_{1,1}(\omega)\mu_\beta(\omega) = 0.$$

By [Spr12a, Remark 2.19], $\det(M_p) = \frac{\log_p(1+X)}{X} \times \frac{\beta-\alpha}{(\alpha\beta)^2}$. But $\alpha \neq \beta$ by [CB98, Theorem 2.1], hence the result. □

Remark 1.6. *Similar logarithmic matrices have been constructed in* [LLZ10] *using the theory of Wach modules, but they are not canonical.*

2. Two-variable p-adic L-functions

2.1. **Setup for two-variable distributions.** We now fix an imaginary quadratic field K in which p splits into $\mathfrak{p}\overline{\mathfrak{p}}$. If $\mathfrak{I}$ is an ideal of K, we write $G_{\mathfrak{I}}$ for the ray class group of K modulo $\mathfrak{I}$. Define

$$G_{p^\infty} = \varprojlim G_{p^n}, \quad G_{\mathfrak{p}^\infty} = \varprojlim G_{\mathfrak{p}^n}, \quad G_{\overline{\mathfrak{p}}^\infty} = \varprojlim G_{\overline{\mathfrak{p}}^n}.$$

These are the Galois groups of the ray class fields $K(p^\infty)$, $K(\mathfrak{p}^\infty)$ and $K(\overline{\mathfrak{p}}^\infty)$ respectively. Fix topological generators $\gamma_{\mathfrak{p}}$ and $\gamma_{\overline{\mathfrak{p}}}$ of the $\mathbb{Z}_p$-parts of $G_{\mathfrak{p}^\infty}$ and $G_{\overline{\mathfrak{p}}^\infty}$ respectively. We have an isomorphism

$$G_{p^\infty} \cong \Delta \times \langle\gamma_{\mathfrak{p}}\rangle \times \langle\gamma_{\overline{\mathfrak{p}}}\rangle,$$

where Δ is a finite abelian group. For real numbers $u, v \geq 0$, we define $D^{(u,v)}(G_{p^\infty}, F)$ to be the set of distributions μ of G_{p^∞} such that for fixed integers $m, n \geq 0$,

$$\inf_{g \in G_{p^\infty}} v_p\left(\mu\left(1_{g\langle\gamma_{\mathfrak{p}}\rangle^{p^m}\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\right)\right) \geq R - um - vn$$

for some constant $R \in \mathbb{R}$ that only depends on μ.

Let $X = \gamma_{\mathfrak{p}} - 1$ and $Y = \gamma_{\overline{\mathfrak{p}}} - 1$. We may identify an element of $D^{(u,v)}(G_{p^\infty}, F)$ with a power series

$$\sum_{i,j \geq 0} \sum_{\sigma \in \Delta} c_{\sigma,i,j} \sigma X^i Y^j,$$

where $c_{\sigma,i,j} \in F$. On identifying each Δ-isotypical component of μ with a power series in X and Y, we have the notion of divisibility as in the one-dimensional case. We define the operator $\partial_{\mathfrak{p}}$ to be the partial derivative $\frac{\partial}{\partial X}$.

For $\star \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, we let $\Omega_\star$ be the set of characters on G_{p^∞} with conductor $(\star)^n$ for some integer $n \geq 1$.

Let $\mu \in D^{(u,v)}(G_{p^\infty}, F)$ where $u, v \geq 0$. If $\omega_{\mathfrak{p}} \in \Omega_{\mathfrak{p}}$, we define a distribution $\mu^{(\omega_{\mathfrak{p}})}$ by

$$\mu^{(\omega_{\mathfrak{p}})}(\omega_{\overline{\mathfrak{p}}}) = \mu(\omega_{\mathfrak{p}}\omega_{\overline{\mathfrak{p}}}).$$

Lemma 2.1. *The distribution $\mu^{(\omega_{\mathfrak{p}})}$ lies inside $D^{(v)}(G_{\overline{\mathfrak{p}}^\infty}, F')$, where F' is the extension $F(\omega_{\mathfrak{p}}(\gamma_{\mathfrak{p}}))$.*

Proof. By definition, for any integers $m, n \geq 0$, we have

$$\inf_{g \in G_{p^\infty}} v_p\left(\mu\left(1_{g\langle\gamma_{\mathfrak{p}}\rangle^{p^m}\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\right)\right) \geq R - um - vn \tag{2}$$

for some R. Since $\omega_{\mathfrak{p}}$ is of finite order, we have

$$\omega_{\mathfrak{p}} = \sum_{h \in G_{p^\infty} / \ker(\omega_{\mathfrak{p}})} \omega_{\mathfrak{p}}(h) 1_{h \ker(\omega_{\mathfrak{p}})}.$$

Moreover,

$$v_p(\omega_{\mathfrak{p}}(h)) = 0 \tag{3}$$

for all h and $\ker(\omega_{\mathfrak{p}}) \cap \langle\gamma_{\mathfrak{p}}\rangle\langle\gamma_{\overline{\mathfrak{p}}}\rangle = \langle\gamma_{\mathfrak{p}}\rangle^{p^m}\langle\gamma_{\overline{\mathfrak{p}}}\rangle$ for some integer m.

If $g \in G_{\overline{\mathfrak{p}}^\infty}$ and $n \geq 0$ is an integer, we may lift the coset $g\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}$ in $G_{\overline{\mathfrak{p}}^\infty}$ to one in G_{p^∞} that is of the form $g'\langle\gamma_{\mathfrak{p}}\rangle\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}$. Therefore,

$$\begin{aligned}
\mu^{(\omega_{\mathfrak{p}})}\left(1_{g\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\right) &= \mu\left(\sum_{h \in G_{p^\infty} / \ker(\omega_{\mathfrak{p}})} \omega_{\mathfrak{p}}(h) 1_{h \ker(\omega_{\mathfrak{p}})} 1_{g'\langle\gamma_{\mathfrak{p}}\rangle\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\right) \\
&= \sum_{h \in G_{p^\infty} / \ker(\omega_{\mathfrak{p}})} \omega_{\mathfrak{p}}(h)\mu\left(1_{h \ker(\omega_{\mathfrak{p}}) \cap g'\langle\gamma_{\mathfrak{p}}\rangle\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\right) \\
&= \sum_{\substack{h \in G_{p^\infty} / \ker(\omega_{\mathfrak{p}}) \\ h \in g'\langle\gamma_{\mathfrak{p}}\rangle\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}} \omega_{\mathfrak{p}}(h)\mu\left(1_{h\langle\gamma_{\mathfrak{p}}\rangle^{p^m}\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\right).
\end{aligned}$$

Therefore, by (2) and (3), we have

$$v_p\left(\mu^{(\omega_{\mathfrak{p}})}\left(1_{g\langle\gamma_{\overline{\mathfrak{p}}}\rangle^n}\right)\right) \geq (R - um) - vn$$

as required. ∎

Similarly, for $\omega_{\overline{\mathfrak{p}}} \in \Omega_{\overline{\mathfrak{p}}}$, we may define a distribution $\mu^{(\omega_{\overline{\mathfrak{p}}})} \in D^{(u)}(G_{\mathfrak{p}^\infty}, F')$, where $F' = F(\omega_{\overline{\mathfrak{p}}}(\gamma_{\overline{\mathfrak{p}}}))$.

2.2. **Sprung-type factorisation.** Let $L_{\alpha,\alpha}$, $L_{\alpha,\beta}$, $L_{\beta,\alpha}$, $L_{\beta,\beta}$ be the two-variable p-adic L-functions constructed in [Loe13] (note that $L_{\alpha,\alpha}$ and $L_{\beta,\beta}$ have also be constructed in [Kim11]). By [Loe13, Theorem 4.7], $L_{\star,\bullet}$ is an element of $D^{(v_p(\star),v_p(\bullet))}(G_{p^\infty}, F)$ for $\star, \bullet \in \{\alpha, \beta\}$. Moreover, if ω is a character on G_{p^∞} of conductor $\mathfrak{p}^{n_\mathfrak{p}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ with $n_\mathfrak{p}, n_{\overline{\mathfrak{p}}} \geq 1$, we have

$$L_{\alpha,\alpha}(\omega) = \alpha^{-n_\mathfrak{p}}\alpha^{-n_{\overline{\mathfrak{p}}}}C_\omega \tag{4}$$
$$L_{\alpha,\beta}(\omega) = \alpha^{-n_\mathfrak{p}}\beta^{-n_{\overline{\mathfrak{p}}}}C_\omega \tag{5}$$
$$L_{\beta,\alpha}(\omega) = \beta^{-n_\mathfrak{p}}\alpha^{-n_{\overline{\mathfrak{p}}}}C_\omega \tag{6}$$
$$L_{\beta,\beta}(\omega) = \beta^{-n_\mathfrak{p}}\beta^{-n_{\overline{\mathfrak{p}}}}C_\omega \tag{7}$$

for some $C_\omega \in \overline{F}$ that is independent of α and β.

Let M_p be the logarithmic matrix given by Theorem 1.5. On replacing γ_p by $\gamma_\mathfrak{p}$ and $\gamma_{\overline{\mathfrak{p}}}$ respectively, we have two logarithmic matrices $M_\mathfrak{p} = \begin{pmatrix} m^\mathfrak{p}_{1,1} & m^\mathfrak{p}_{1,2} \\ m^\mathfrak{p}_{2,1} & m^\mathfrak{p}_{2,2}\end{pmatrix}$ and $M_{\overline{\mathfrak{p}}} = \begin{pmatrix} m^{\overline{\mathfrak{p}}}_{1,1} & m^{\overline{\mathfrak{p}}}_{1,2} \\ m^{\overline{\mathfrak{p}}}_{2,1} & m^{\overline{\mathfrak{p}}}_{2,2}\end{pmatrix}$ defined over $D^{(1)}(\langle\gamma_\mathfrak{p}\rangle, F)$ and $D^{(1)}(\langle\gamma_{\overline{\mathfrak{p}}}\rangle, F)$ respectively.

Our goal is to prove the following generalisation of [Loe13, Corollary 5.4].

Theorem 2.2. *There exist $L_{\#,\#}, L_{\flat,\#}, L_{\#,\flat}, L_{\flat,\flat} \in D^{(0,0)}(G_{p^\infty}, F)$ such that*

$$\begin{pmatrix} L_{\alpha,\alpha} & L_{\beta,\alpha} & L_{\alpha,\beta} & L_{\beta,\beta}\end{pmatrix} = \begin{pmatrix} L_{\#,\#} & L_{\flat,\#} & L_{\#,\flat} & L_{\flat,\flat}\end{pmatrix} M_\mathfrak{p} \otimes M_{\overline{\mathfrak{p}}}.$$

We shall prove this theorem in two steps, namely, to show that we can first factor out $M_\mathfrak{p}$, then $M_{\overline{\mathfrak{p}}}$.

Proposition 2.3. *For $\star \in \{\alpha, \beta\}$, there exist $L_{\#,\star}, L_{\flat,\star} \in D^{(0,v_p(\star))}(G_{p^\infty}, F)$ such that*

$$\begin{pmatrix} L_{\alpha,\star} & L_{\beta,\star}\end{pmatrix} = \begin{pmatrix} L_{\#,\star} & L_{\flat,\star}\end{pmatrix} M_\mathfrak{p}. \tag{8}$$

Proof. We take $\star = \alpha$ (since the proof for the case $\star = \beta$ is identical). Let $\omega_\mathfrak{p} \in \Omega_\mathfrak{p}$ and $\omega_{\overline{\mathfrak{p}}} \in \Omega_{\overline{\mathfrak{p}}}$ and write $\omega = \omega_\mathfrak{p}\omega_{\overline{\mathfrak{p}}}$.

By (4) and (6), $L^{(\omega_{\overline{\mathfrak{p}}})}_{\alpha,\alpha}$ and $L^{(\omega_{\overline{\mathfrak{p}}})}_{\beta,\alpha}$ is a pair of interpolating functions for f. In particular, $\det(M_\mathfrak{p})$ divides both

$$m^\mathfrak{p}_{2,2}L^{(\omega_{\overline{\mathfrak{p}}})}_{\alpha,\alpha} - m^\mathfrak{p}_{2,1}L^{(\omega_{\overline{\mathfrak{p}}})}_{\beta,\alpha} \quad \text{and} \quad -m^\mathfrak{p}_{1,2}L^{(\omega_{\overline{\mathfrak{p}}})}_{\alpha,\alpha} + m^\mathfrak{p}_{1,1}L^{(\omega_{\overline{\mathfrak{p}}})}_{\beta,\alpha}$$

over $D^{(1)}(G_{\mathfrak{p}^\infty}, F)$. Therefore, the distributions

$$m^\mathfrak{p}_{2,2}L_{\alpha,\alpha} - m^\mathfrak{p}_{2,1}L_{\beta,\alpha} \quad \text{and} \quad -m^\mathfrak{p}_{1,2}L_{\alpha,\alpha} + m^\mathfrak{p}_{1,1}L_{\beta,\alpha}$$

vanish at all characters of the form $\omega = \omega_\mathfrak{p}\omega_{\overline{\mathfrak{p}}}$. This implies that

$$\left(m^\mathfrak{p}_{2,2}L_{\alpha,\alpha} - m^\mathfrak{p}_{2,1}L_{\beta,\alpha}\right)^{(\omega_\mathfrak{p})} = \left(-m^\mathfrak{p}_{1,2}L_{\alpha,\alpha} + m^\mathfrak{p}_{1,1}L_{\beta,\alpha}\right)^{(\omega_\mathfrak{p})} = 0$$

since these two distributions lie inside $D^{(r)}(G_{\overline{\mathfrak{p}}^\infty}, F')$ for some F', with $r < 1$, and they vanish at an infinite number of characters for each of their isotypical components. Hence, $\det(M_\mathfrak{p})$ divides

$$m^\mathfrak{p}_{2,2}L_{\alpha,\alpha} - m^\mathfrak{p}_{2,1}L_{\beta,\alpha} \quad \text{and} \quad -m^\mathfrak{p}_{1,2}L_{\alpha,\alpha} + m^\mathfrak{p}_{1,1}L_{\beta,\alpha}$$

over $D^{(1,r)}(G_{p^\infty},F)$. Let

$$L_{\#,\alpha}:=\frac{m_{2,2}^{\mathfrak{p}}L_{\alpha,\alpha}-m_{2,1}^{\mathfrak{p}}L_{\beta,\alpha}}{\det(M_{\mathfrak{p}})}\quad\text{and}\quad L_{\flat,\alpha}:=\frac{-m_{1,2}^{\mathfrak{p}}L_{\alpha,\alpha}+m_{1,1}^{\mathfrak{p}}L_{\beta,\alpha}}{\det(M_{\mathfrak{p}})}.$$

We may then conclude as in the proof of Lemma 1.4. □

Lemma 2.4. *Let ω be a character of G_{p^∞} of conductor $\mathfrak{p}^{n_{\mathfrak{p}}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ with $n_{\mathfrak{p}},n_{\overline{\mathfrak{p}}}\geq 1$. There exist constants D_ω and E_ω in $\overline{F}$ such that*

$$\partial_{\mathfrak{p}}L_{\alpha,\alpha}(\omega)=\alpha^{-n_{\overline{\mathfrak{p}}}}D_\omega,\qquad \partial_{\mathfrak{p}}L_{\alpha,\beta}(\omega)=\beta^{-n_{\overline{\mathfrak{p}}}}D_\omega,$$
$$\partial_{\mathfrak{p}}L_{\beta,\alpha}(\omega)=\alpha^{-n_{\overline{\mathfrak{p}}}}E_\omega,\qquad \partial_{\mathfrak{p}}L_{\beta,\beta}(\omega)=\beta^{-n_{\overline{\mathfrak{p}}}}E_\omega.$$

Proof. We only prove the result concerning $\partial_{\mathfrak{p}}L_{\alpha,\alpha}$ and $\partial_{\mathfrak{p}}L_{\alpha,\beta}$. Fix an $\omega_{\overline{\mathfrak{p}}}\in\Omega_{\overline{\mathfrak{p}}}$. By (8) and (9), we have

$$\beta^{n_{\overline{\mathfrak{p}}}}L_{\alpha,\beta}^{(\omega_{\overline{\mathfrak{p}}})}(\omega_{\mathfrak{p}})=\alpha^{n_{\overline{\mathfrak{p}}}}L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})}(\omega_{\mathfrak{p}})$$

for all $\omega_{\mathfrak{p}}\in\Omega_{\mathfrak{p}}$. But $L_{\alpha,\beta}^{(\omega_{\overline{\mathfrak{p}}})},L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})}\in D^{(r)}(G_{\mathfrak{p}^\infty},F')$ for some F'. As $r<1$, this implies that

$$\beta^{n_{\overline{\mathfrak{p}}}}L_{\alpha,\beta}^{(\omega_{\overline{\mathfrak{p}}})}=\alpha^{n_{\overline{\mathfrak{p}}}}L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})}.$$

In particular, their derivatives agree, that is

$$\beta^{n_{\overline{\mathfrak{p}}}}\partial_{\mathfrak{p}}L_{\alpha,\beta}^{(\omega_{\overline{\mathfrak{p}}})}=\alpha^{n_{\overline{\mathfrak{p}}}}\partial_{\mathfrak{p}}L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})}.$$

But for a general $\mu\in D^{(r,s)}(G_{p^\infty},F)$, we have

$$\partial_{\mathfrak{p}}\left(\mu^{(\omega_{\overline{\mathfrak{p}}})}\right)(\omega_{\mathfrak{p}})=\partial_{\mathfrak{p}}\mu(\omega_{\mathfrak{p}}\omega_{\overline{\mathfrak{p}}})$$

for all $\omega_{\mathfrak{p}}\in\Omega_{\mathfrak{p}}$, hence

$$\beta^{n_{\overline{\mathfrak{p}}}}\partial_{\mathfrak{p}}L_{\alpha,\beta}(\omega)=\alpha^{n_{\overline{\mathfrak{p}}}}\partial_{\mathfrak{p}}L_{\alpha,\alpha}(\omega)$$

as required. □

Proposition 2.5. *For $\star\in\{\#,\flat\}$, there exist $L_{\star,\#},L_{\star,\flat}\in D^{(0,0)}(G_{p^\infty},F)$ such that*

$$\begin{pmatrix}L_{\star,\alpha} & L_{\star,\beta}\end{pmatrix}=\begin{pmatrix}L_{\star,\#} & L_{\star,\flat}\end{pmatrix}M_{\overline{\mathfrak{p}}}.\tag{9}$$

Proof. Let us prove the proposition for $\star=\#$. Let $\omega_{\mathfrak{p}}\in\Omega_{\mathfrak{p}}$ and $\omega_{\overline{\mathfrak{p}}}\in\Omega_{\overline{\mathfrak{p}}}$ and write $\omega=\omega_{\mathfrak{p}}\omega_{\overline{\mathfrak{p}}}$. Recall that

$$L_{\#,\bullet}\det(M_{\mathfrak{p}})=m_{2,2}^{\mathfrak{p}}L_{\alpha,\bullet}-m_{2,1}^{\mathfrak{p}}L_{\beta,\bullet}$$

for $\bullet\in\{\alpha,\beta\}$. Since $\det(M_{\mathfrak{p}})$ is, up to a non-zero constant in $F^\times$, equal to $\log_p(1+X)/X$, we have $\det(M_p)(\omega_{\mathfrak{p}})=0$ and $\partial_{\mathfrak{p}}\det(M_{\mathfrak{p}})(\omega_{\mathfrak{p}})\neq 0$. On taking partial derivatives, Lemma 2.4 together with (4)-(7) imply that

$$L_{\#,\bullet}(\omega)=(\bullet)^{-n_{\overline{\mathfrak{p}}}}\frac{K_\omega}{\partial_{\mathfrak{p}}\det(M_{\mathfrak{p}})(\omega_{\mathfrak{p}})},$$

where K_ω is the constant

$$m_{2,2}^{\mathfrak{p}}(\omega_{\mathfrak{p}})D_\omega+\partial_{\mathfrak{p}}m_{2,2}^{\mathfrak{p}}(\omega_{\mathfrak{p}})\alpha^{-n_{\mathfrak{p}}}C_\omega-m_{2,1}^{\mathfrak{p}}(\omega_{\mathfrak{p}})E_\omega-\partial_{\mathfrak{p}}m_{2,1}^{\mathfrak{p}}(\omega_{\mathfrak{p}})\beta^{-n_{\mathfrak{p}}}C_\omega.$$

In particular, we see that $L_{\#,\alpha}^{(\omega_{\mathfrak{p}})}$ and $L_{\#,\beta}^{(\omega_{\mathfrak{p}})}$ is a pair of interpolating functions for f, so we may proceed as in the proof of Proposition 2.3 (with the roles of $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ swapped). □

Combining the factorisations (8) and (9), we obtain Theorem 2.2. Note that our proof is very different from that of [Loe13, Corollary 5.4]. In fact, it only relies on the properties of logarithmic matrices as specified in Definition 1.3. Therefore, if we replace M_p by *any* logarithmic matrices, Theorem 2.2 still holds. For example, one may take M_p to be the non-canonical logarithmic matrices mentioned in Remark 1.6.

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