IWASAWA THEORY FOR THE SYMMETRIC SQUARE OF A CM MODULAR FORM AT INERT PRIMES

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Abstract. Let f be a CM modular form and p an odd prime which is inert in the CM field. We construct two p-adic L-functions for the symmetric square of f, one of which has the same interpolating properties as the one constructed by Delbourgo-Dabrowski, whereas the second one has a similar interpolating properties but corresponds to a different eigenvalue of the Frobenius. The symmetry between these two p-adic L-functions allows us to define the plus and minus p-adic L-functions à la Pollack. We also define the plus and minus p-Selmer groups analogous to Kobayashi's Selmer groups. We explain how to relate these two sets of objects via a main conjecture.

1. Introduction

Let f be a normalised eigen-newform of weight k, level N and character ϵ. Fix a prime $p \neq 2$ such that $p \nmid N$. In [3] (also in [1] under some additional conditions), even distributions on $\mathbb{Z}_p^\times$ are constructed to interpolate the L-values of the symmetric square of f. More precisely, if the Euler factor of $L(E,s)$ at p is given by $(1-\alpha_1(p)p^{-s})(1-\alpha_2(p)p^{-s})$, then there exists an admissible distribution $\mu_{\alpha_i(p)^2}$ for $i=1,2$ such that

$$\int_{\mathbb{Z}_p^\times} \theta d\mu_{\alpha_i(p)^2} = \frac{p^{3n(k-1)}}{\alpha_i(p)^{2n}\tau(\theta^{-1})} \times \frac{L(\mathrm{Sym}^2 f, \theta^{-1}, 2k-2)}{(\text{period})} \tag{1}$$

for any non-trivial even Dirichlet character θ of conductor p^n where $\tau(\theta^{-1})$ denotes the Gauss sum of θ^{-1}.

Since the Euler factor of $L(\mathrm{Sym}^2 f, s)$ at p is $(1-\alpha_1(p)^2p^{-s})(1-\alpha_2(p)^2p^{-s})(1-\epsilon(p)p^{k-1-s})$, we expect that there should be a distribution $\mu_{\epsilon(p)p^{k-1}}$ satisfying interpolating properties similar to (1), but with $\alpha_i(p)^2$ replaced by $\epsilon(p)p^{k-1}$. In this paper, we construct such a distribution for the case when f is a CM modular form that is non-ordinary at p. In other words, when the L-function of f coincides with that of a Grossencharacter ϕ defined over K and p inerts in K. More precisely, we prove the following theorem in §3 (Theorem 3.20).

Theorem 1.1. *If f is as above, then there exist even admissible distributions $\mu_{\pm\epsilon(p)p^{k-1}}$ such that*

$$\int_{\mathbb{Z}_p^\times} \theta d\mu_{\pm\epsilon(p)p^{k-1}} = \frac{p^{3n(k-1)}}{\left(\pm\epsilon(p)p^{k-1}\right)^n \tau(\theta^{-1})} \times \frac{L(\mathrm{Sym}^2 f, \theta^{-1}, 2k-2)}{(\text{period})}.$$

Note that we have $\alpha_1(p)^2 = \alpha_2(p)^2 = -\epsilon(p)p^{k-1}$ in this case, methods in [3] only produce one distribution, which agrees with $\mu_{-\epsilon(p)p^{k-1}}$ as given by Theorem 1.1.

The idea of the construction is rather simple. Let V_f be the p-adic representation of $G_{\mathbb{Q}}$ associated to f as constructed by Deligne in [4]. In order to prove Theorem 1.1, we make use of the following observation. As $G_{\mathbb{Q}}$-representations, we have

$$\mathrm{Sym}^2(V_f) \cong V_1 \oplus V_2$$

where V_1 is an one-dimensional representation associated to some Dirichlet character η twisted by a power of the cyclotomic character and V_2 is a two-dimensional representation associated to the Grossencharacter ϕ^2. This implies that the L-function of f factorises into

$$L(\mathrm{Sym}^2 f, s) = L(\phi^2, s)L(\eta, s-k+1).$$

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We can therefore make use of an Euler system constructed from elliptic units to interpolate the L-values of ϕ^2 and multiply the resulting distributions with an appropriate twist of the Kubota-Leopoldt p-adic L-function associated to η, which interpolates the L-values of η.

Because of the symmetry between the two distributions, we show that some plus and minus logarithms $\log^{\pm}$ of Pollack divide $\mu_{+\epsilon(p)p^{k-1}} \pm \mu_{-\epsilon(p)p^{k-1}}$. This allows us to obtain two bounded measures:

Theorem 1.2. *(Theorem 3.25) Let θ be an even Dirichlet character of conductor p^n. There exist bounded p-adic measures $\mu^{\pm}(\mathrm{Sym}^2(V_f))$ such that the followings hold.*

(a) *If n is even, then*

$$\int_{\mathbb{Z}_p^\times} \theta\mu^+(\mathrm{Sym}^2(V_f)) = \frac{(2k-3)!(k-1)!p^{2n(k-1)}}{\theta\left(\log^+\right)\tau(\theta^{-1})^2\epsilon(p)^n} \times \frac{L(\mathrm{Sym}^2 f, \theta^{-1}, 2k-2)}{(\text{period})};$$

(b) *If n is odd, then*

$$\int_{\mathbb{Z}_p^\times} \theta\mu^-(\mathrm{Sym}^2(V_f)) = \frac{(2k-3)!(k-1)!p^{2n(k-1)}}{\theta\left(\log^-\right)\tau(\theta^{-1})^2\epsilon(p)^n} \times \frac{L(\mathrm{Sym}^2 f, \theta^{-1}, 2k-2)}{(\text{period})}.$$

Moreover, $\mu^{\pm}(\mathrm{Sym}^2(V_f))$ are uniquely determined by (a) and (b) respectively.

In § 4, we make use of some of the ideas in [6] to show that these measures can be obtained from some appropriate Coleman maps and define the corresponding plus and minus p-Selmer groups $\mathrm{Sel}_p^{\pm}(\mathrm{Sym}^2(V_f))$. On identifying the measures as elements in some Iwasawa algebra $\Lambda \otimes \mathbb{Q}$, we show that the following holds under some appropriate conditions (see Theorem 4.8 for a precise statement).

Theorem 1.3. *The Selmer groups $\mathrm{Sel}_p^{\pm}(\mathrm{Sym}^2(V_f))$ are Λ-cotorsion and*

$$\mathrm{Char}_{\Lambda\otimes\mathbb{Q}}\left(\mathrm{Sel}_p^{\pm}(\mathrm{Sym}^2(V_f))^\vee\right) = \left(\mu^{\pm}(\mathrm{Sym}^2(V_f))\right).$$

Finally, in the appendix, we explain how some of the linear algebra results that we use to prove the main theorems can be easily generalised to general symmetric powers $\mathrm{Sym}^m f$ where $m \geq 2$ is an integer.

2. Notation

2.1. **Extensions by p power roots of unity.** Throughout this paper, p is an odd prime. If K is a field of characteristic 0, either local or global, G_K denotes its absolute Galois group, χ the p-cyclotomic character on G_K and $\mathcal{O}_K$ the ring of integers of K. We write ι for the complex conjugation in $G_{\mathbb{Q}}$.

For an integer $n \geq 0$, we write K_n for the extension $K(\mu_{p^n})$ where μ_{p^n} is the set of p^nth roots of unity and K_∞ denotes $\bigcup_{n\geq 1} K_n$. When $K = \mathbb{Q}$, we write $k_n = \mathbb{Q}(\mu_{p^n})$ instead. In particular, we write $\mathbb{Q}_{p,n} = \mathbb{Q}_p(\mu_{p^n})$. Let G_n denote the Galois group $\mathrm{Gal}(\mathbb{Q}_{p,n}/\mathbb{Q}_p)$ for $0 \leq n \leq \infty$. Then, $G_\infty \cong \Delta \times \Gamma$ where $\Delta = G_1$ is a finite group of order $p-1$ and $\Gamma = \mathrm{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_{p,1}) \cong \mathbb{Z}_p$. We fix a topological generator γ of Γ.

2.2. **Iwasawa algebras and power series.** Given a finite extension K of $\mathbb{Q}_p$, $\Lambda_{\mathcal{O}_K}(G_\infty)$ (respectively $\Lambda_{\mathcal{O}_K}(\Gamma)$) denotes the Iwasawa algebra of G_∞ (respectively Γ) over $\mathcal{O}_K$. We write $\Lambda_K(G_\infty) = \Lambda_{\mathcal{O}_K}(G_\infty) \otimes K$ and $\Lambda_K(\Gamma) = \Lambda_{\mathcal{O}_K}(\Gamma) \otimes K$. If M is a finitely generated $\Lambda_{\mathcal{O}_K}(\Gamma)$-torsion (respectively $\Lambda_K(\Gamma)$-torsion) module, we write $\mathrm{Char}_{\Lambda_{\mathcal{O}_K}(\Gamma)}(M)$ (respectively $\mathrm{Char}_{\Lambda_K(\Gamma)}(M)$) for its characteristic ideal.

Given a module M over $\Lambda_{\mathcal{O}_K}(G_\infty)$ (respectively $\Lambda_K(G_\infty)$) and a character $\delta : \Delta \to \mathbb{Z}_p^\times$, M^δ denotes the δ-isotypical component of M. For any $m \in M$, we write m^δ for the projection of m into M^δ. The Pontryagin dual of M is written as $M^\vee$.

Let $r \in \mathbb{R}_{\geq 0}$. We define

$$\mathcal{H}_r = \left\{ \sum_{n\geq 0, \sigma\in\Delta} c_{n,\sigma} \cdot \sigma \cdot X^n \in \mathbb{C}_p[\Delta][[X]] : \sup_n \frac{|c_{n,\sigma}|_p}{n^r} < \infty \ \forall \sigma \in \Delta \right\}$$

where $|\cdot|_p$ is the p-adic norm on $\mathbb{C}_p$ such that $|p|_p = p^{-1}$. We write $\mathcal{H}_\infty = \cup_{r\geq 0}\mathcal{H}_r$ and $\mathcal{H}_r(G_\infty) = \{f(\gamma - 1) : f \in \mathcal{H}_r\}$ for $r \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. In other words, the elements of $\mathcal{H}_r$ (respectively $\mathcal{H}_r(G_\infty)$) are the power series in X (respectively $\gamma - 1$) over $\mathbb{C}_p[\Delta]$ with growth rate $O(\log_p^r)$. If $F, G \in \mathcal{H}_\infty$ or $\mathcal{H}_\infty(G_\infty)$ are such that $F = O(G)$ and $G = O(F)$, we write $F \sim G$.

Given a subfield K of $\mathbb{C}_p$, we write $\mathcal{H}_{r,K} = \mathcal{H}_r \cap K[\Delta][[X]]$ and similarly for $\mathcal{H}_{r,K}(G_\infty)$. In particular, $\mathcal{H}_{0,K}(G_\infty) = \Lambda_K(G_\infty)$.

Let $n \in \mathbb{Z}$. We define the K-linear map Tw_n from $\mathcal{H}_{r,K}(G_\infty)$ to itself to be the map that sends σ to $\chi(\sigma)^n\sigma$ for all $\sigma \in G_\infty$. It is clearly bijective (with inverse Tw_{-n}).

2.3. **Crystalline representations.** We write $\mathbb{B}_{\mathrm{cris}}$ and $\mathbb{B}_{\mathrm{dR}}$ for the rings of Fontaine and φ for the Frobenius acting on these rings. Recall that there exists an element $t \in \mathbb{B}_{\mathrm{dR}}$ such that $\varphi(t) = pt$ and $g \cdot t = \chi(g)t$ for $g \in G_{\mathbb{Q}_p}$.

Let V be a p-adic representation of $G_{\mathbb{Q}_p}$. We denote the Dieudonné module by $\mathbb{D}_{\mathrm{cris}}(V) = (\mathbb{B}_{\mathrm{cris}} \otimes V)^{G_{\mathbb{Q}_p}}$. We say that V is crystalline if V has the same $\mathbb{Q}_p$-dimension as $\mathbb{D}_{\mathrm{cris}}(V)$. Fix such a V. If $j \in \mathbb{Z}$, $\mathrm{Fil}^j\, \mathbb{D}_{\mathrm{cris}}(V)$ denotes the jth de Rham filtration of $\mathbb{D}_{\mathrm{cris}}(V)$.

Let T be a lattice of V which is stable under $G_{\mathbb{Q}_p}$. Let $\mathbb{H}^1_{\mathrm{Iw}}(T)$ denote the inverse limit $\varprojlim H^1(\mathbb{Q}_{p,n}, T)$ with respect to the corestriction and $\mathbb{H}^1_{\mathrm{Iw}}(V) = \mathbb{Q} \otimes \mathbb{H}^1_{\mathrm{Iw}}(T)$. Moreover, if V arises from the restriction of a p-adic representation of $G_{\mathbb{Q}}$ and T is a lattice stable under $G_{\mathbb{Q}}$, we write

$$\mathbb{H}^1(T) = \varprojlim_n H^1(\mathbb{Z}[\mu_{p^n}, 1/p], T) \quad \text{and} \quad \mathbb{H}^1(V) = \mathbb{Q} \otimes \mathbb{H}^1(T).$$

We have localisation maps

$$\mathrm{loc} : \mathbb{H}^1(T) \to \mathbb{H}^1_{\mathrm{Iw}}(T) \quad \text{and} \quad \mathrm{loc} : \mathbb{H}^1(V) \to \mathbb{H}^1_{\mathrm{Iw}}(V).$$

If F is a number field, we define the p-Selmer group of T over F to by

$$\mathrm{Sel}_p(T/F) = \ker\left(H^1(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \to \prod_v \frac{H^1(F_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H^1_f(F_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)} \right)$$

where v runs through the places of F.

Let $V(j)$ denote the jth Tate twist of V, i.e. $V(j) = V \otimes \mathbb{Q}_p e_j$ where $G_{\mathbb{Q}_p}$ acts on e_j via χ^j. We have

$$\mathbb{D}_{\mathrm{cris}}(V(j)) = t^{-j}\mathbb{D}_{\mathrm{cris}}(V) \otimes e_j.$$

For any $v \in \mathbb{D}_{\mathrm{cris}}(V)$, $v_j = v \otimes t^{-j}e_j$ denotes its image in $\mathbb{D}_{\mathrm{cris}}(V(j))$. We write $\mathrm{Tw}_j : \mathbb{H}^1_{\mathrm{Iw}}(V) \to \mathbb{H}^1_{\mathrm{Iw}}(V(j))$ for the isomorphism defined in [9, § A.4], which depends on a choice of primitive p-power roots of unity.

Finally, we write

$$\exp : \mathbb{Q}_{p,n} \otimes \mathbb{D}_{\mathrm{cris}}(V) \to H^1(\mathbb{Q}_{p,n}, V) \quad \text{and} \quad \exp^* : H^1(\mathbb{Q}_{p,n}, V) \to \mathbb{Q}_{p,n} \otimes \mathrm{Fil}^0\, \mathbb{D}_{\mathrm{cris}}(V)$$

for Bloch-Kato's exponential and dual exponential respectively.

2.4. **Imaginary quadratic fields.** Let K be an imaginary quadratic field with ring of integers $\mathcal{O}$ and idele class group C_K. We write ε_K for the quadratic character associated to K, i.e. the character on $G_{\mathbb{Q}}$ which sends σ to 1 if $\sigma \in G_K$ and to -1 otherwise.

A Grossencharacter of K is simply a continuous homomorphism $\phi : C_K \to \mathbb{C}^\times$ with complex L-function

$$L(\phi, s) = \prod_v (1 - \phi(v)N(v)^{-s})^{-1}$$

where the product runs through the finite places v of K at which ϕ is unramified, $\phi(v)$ is the image of the uniformiser of K_v under ϕ and $N(v)$ is the norm of v. Let $\mathfrak{f}$ be the conductor of ϕ. We say that η is of type (m, n) where $m, n \in \mathbb{Z}$ if the restriction of η to the archimedean part $\mathbb{C}^\times$ of C_K is of the form $z \mapsto z^m \bar{z}^n$.

We write $\mathcal{K} = \cup K(p^n\mathfrak{f})$ where $K(\mathfrak{a})$ denotes the ray class field of K modulo $\mathfrak{a}$ if $\mathfrak{a}$ is an ideal of $\mathcal{O}$.

If T is a $\mathbb{Z}_p$-representation of G_K, we write

$$\mathbb{H}^1_{p^\infty\mathfrak{f}}(T) = \varprojlim_{K'} H^1(\mathcal{O}_{K'}[1/p], T) \quad \text{and} \quad \mathbb{H}^1_{p^\infty\mathfrak{f}}(\mathbb{Q} \otimes_{\mathbb{Z}_p} T) = \mathbb{H}^1_{p^\infty\mathfrak{f}}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}$$

where K' ranges over all finite extensions of K contained in $K(p^\infty\mathfrak{f})$.

2.5. **Modular forms.** Let $f = \sum a_n q^n$ be a normalised eigen-newform of weight $k \geq 2$, level N and character ϵ. We assume that f is a CM modular form, i.e. $L(f,s) = L(\phi,s)$ for some Grossencharacter ϕ of an imaginary quadratic field K with conductor $\mathfrak{f}$. Then, ϕ is of type $(-k+1,0)$. Moreover, p inerts in K if and only if f is non-ordinary at p. In this case, a_p is always 0. Throughout, we fix such a p with $p \neq 2$.

The coefficient field F_f of f is contained in the field of definition of ϕ. We write E for the completion of this field at a fixed prime above p.

We write V_f for the 2-dimensional E-linear representation of $G_{\mathbb{Q}}$ associated to f from [4], so we have a homomorphism

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}(V_f).$$

Throughout the paper, we assume that the following hypothesis holds.

Hypothesis 2.1. *If ϵ and K are as above, then $\varepsilon_K \neq \epsilon$.*

3. p-adic L-functions

3.1. **Grossencharacters over K.** We first review some results on Grossencharacters. Let η be a Grossencharacter on G_K of conductor $\mathfrak{f}$. We fix a finite extension E of $\mathbb{Q}_p$ such that E contains the image of η. We write $V(\eta)$ for the one-dimensional E-linear representation of G_K. It is a representation that factors through $\mathrm{Gal}(\mathcal{K}/K)$. For an ideal $\mathfrak{a}$ of $\mathcal{O}$ which is prime to $p\mathfrak{f}$, the Artin symbol $(\mathfrak{a}, \mathcal{K}/K) \in \mathrm{Gal}(\mathcal{K}/K)$ acts on $V(\eta)$ as the multiplication by $\eta(\mathfrak{a})^{-1}$. We write $\tilde{\eta} : G_K \to E^\times$ for the corresponding character.

We write $\tilde{V}_\eta = \mathrm{Ind}_K^{\mathbb{Q}}(V(\eta))$. The canonical homomorphism $K \otimes \mathbb{Q}(\zeta_{p^\infty}) \to K(p^\infty \mathfrak{f})$ induces a map

$$\mathrm{Ind} : \mathbb{H}^1_{p^\infty \mathfrak{f}}(V(\eta)) \to \mathbb{H}^1(\tilde{V}_\eta).$$

Let γ be a non-zero element of $V(\eta)$. By [5, §15.5], a system of norm compatible elliptic units in $K(p^n\mathfrak{f})$ defines an element $z_{p^\infty \mathfrak{f}} \in \mathbb{H}^1_{p^\infty \mathfrak{f}}(\mathbb{Z}_p(1))$. We write the image of $z_{p^\infty \mathfrak{f}}$ under the composition

$$\mathbb{H}^1_{p^\infty \mathfrak{f}}(\mathbb{Z}_p(1)) \xrightarrow{\gamma} \mathbb{H}^1_{p^\infty \mathfrak{f}}(V(\eta)(1)) \xrightarrow{\mathrm{Ind}} \mathbb{H}^1(\tilde{V}_\eta(1)) \xrightarrow{\mathrm{loc}} \mathbb{H}^1_{\mathrm{Iw}}(\tilde{V}_\eta(1)) \xrightarrow{\mathrm{Tw}_{-1}} \mathbb{H}^1_{\mathrm{Iw}}(\tilde{V}_\eta)$$

as $z_\gamma(\eta) = z(\eta)$ and its projection into $H^1(\mathbb{Q}_{p,n}, \tilde{V}_\eta(j))$ is denoted by $z_{j,n}(\eta)$.

Note that the eigenvalues of ι on $\tilde{V}_\eta$ are ± 1, each with multiplicity 1. If $v \in \tilde{V}_\eta$, we write $v^\pm$ for the projection of v into the ± 1-eigenspace.

Proposition 3.1. *Let η be a Grossencharacter over K of type $(-r,0)$ with $r \geq 1$. Let θ be a character on G_n and write*

$$\begin{aligned} \kappa_\theta : \mathbb{Q}_{p,n} \otimes \mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(\tilde{V}_\eta(1)) &\to \mathbb{C} \otimes \tilde{V}_\eta(1) \\ x \otimes y &\mapsto \sum_{\sigma \in G_n} \theta(\sigma)\sigma(x)\,\mathrm{per}(y) \end{aligned}$$

where per *is the period map associated to η as defined in* [5, §15.8]*. Then, we have*

$$\kappa_\theta \circ \exp^*(z_{1,n}(\eta)) = L_{\{p\}}(\bar{\eta}\theta, r) \cdot (\gamma')^\pm$$

where $\pm = \theta(-1)$ and γ' denotes the image of γ in $\tilde{V}_\eta$.

Proof. [5, §15.12]. □

3.2. **The symmetric square of a CM modular form.** Let f be a modular form as in §2.5. By comparing the eigenvalues of Frobenii, we see that the representation V_f is isomorphic to $\tilde{V}_\phi = \mathrm{Ind}_K^{\mathbb{Q}} V(\phi)$. Therefore, V_f admits a basis x, y such that for $\sigma \in G_{\mathbb{Q}}$, the matrix of $\rho_f(\sigma)$ with respect to this basis is given by

$$\rho_f(\sigma) = \begin{pmatrix} \tilde{\phi}(\sigma) & 0 \\ 0 & \tilde{\phi}(\iota\sigma\iota) \end{pmatrix} \tag{2}$$

if $\sigma \in G_K$. Otherwise,

$$\rho_f(\sigma) = \begin{pmatrix} 0 & \tilde{\phi}(\iota\sigma'\iota) \\ \tilde{\phi}(\sigma') & 0 \end{pmatrix} \tag{3}$$

where $\sigma = \iota\sigma'$ with $\sigma' \in G_K$.

Lemma 3.2. *The determinant of ρ_f is given by*

$$\det(\rho_f)(\sigma) = \begin{cases} \tilde{\phi}(\sigma)\tilde{\phi}(\iota\sigma\iota) & \text{if } \sigma \in G_K \\ -\tilde{\phi}(\sigma')\tilde{\phi}(\iota\sigma'\iota) & \text{if } \sigma = \iota\sigma' \text{ where } \sigma' \in G_K. \end{cases}$$

Proof. This is immediate from (2) and (3). □

Proposition 3.3. *As a $G_{\mathbb{Q}}$-representation, $\mathrm{Sym}^2(V_f)$ decomposes into*

$$\mathrm{Sym}^2(V_f) \cong V_1 \oplus V_2$$

where $\rho_i : G_{\mathbb{Q}} \to GL(V_i)$ is an i-dimensional representation of $G_{\mathbb{Q}}$ for $i = 1, 2$. Moreover,

$$\rho_1 \cong \varepsilon_K \cdot \det(\rho_f) = \varepsilon_K \cdot \epsilon \cdot \chi^{k-1}, \tag{4}$$

$$\rho_2 \cong \tilde{V}_{\phi^2}. \tag{5}$$

Proof. It is clear that $x \otimes x$, $y \otimes y$, $x \otimes y + y \otimes x$ form a basis of $\mathrm{Sym}^2(V_f)$. By formulae (2) and (3), $\sigma \cdot (x \otimes y + y \otimes x)$ is a multiple of $x \otimes y + y \otimes x$ for any $\sigma \in G_{\mathbb{Q}}$. Hence, it gives an one-dimensional sub-representation V_1 of $\mathrm{Sym}^2(V_f)$. More explicitly, we have

$$\sigma \cdot (x \otimes y + y \otimes x) = \begin{cases} \tilde{\phi}(\sigma)\tilde{\phi}(\iota\sigma\iota)(x \otimes y + y \otimes x) & \text{if } \sigma \in G_K \\ \tilde{\phi}(\sigma')\tilde{\phi}(\iota\sigma'\iota)(x \otimes y + y \otimes x) & \text{if } \sigma = \iota\sigma' \text{ where } \sigma' \in G_K. \end{cases}$$

Therefore, we deduce (4) from Lemma 3.2.

It is also clear that $x \otimes x$, $y \otimes y$ form a basis of a 2-dimensional representation $\rho_2 : G_{\mathbb{Q}} \to \mathrm{GL}(V_2)$. With respect to this basis,

$$\rho_2(\sigma) = \begin{pmatrix} \tilde{\phi}^2(\sigma) & 0 \\ 0 & \tilde{\phi}^2(\iota\sigma\iota) \end{pmatrix}$$

if $\sigma \in G_K$. Otherwise, if $\sigma = \iota\sigma'$ where $\sigma' \in G_K$, then

$$\rho_2(\sigma) = \begin{pmatrix} 0 & \tilde{\phi}^2(\iota\sigma'\iota) \\ \tilde{\phi}^2(\sigma') & 0 \end{pmatrix}.$$

Therefore, $V_2 \cong \mathrm{Ind}_K^{\mathbb{Q}} V(\phi^2)$ as required. □

Corollary 3.4. *The complex L function admits a factorisation*

$$L(\mathrm{Sym}^2 f, s) = L(\phi^2, s)L(\varepsilon_K \cdot \epsilon, s - k + 1).$$

Proof. The L-function of $\mathrm{Sym}^2 f$ only have non-trivial Euler factors at $q \nmid N$. The Euler factors on the two sides of the equation at q agree by Proposition 3.3, so we are done. □

3.3. **The symmetric square as a $G_{\mathbb{Q}_p}$-representation.** We study the representation $\mathrm{Sym}^2(V_f)$ restricted to $G_{\mathbb{Q}_p}$. More specifically, we study $\mathbb{D}_{\mathrm{cris}}(\mathrm{Sym}^2 V_f)$.

Lemma 3.5. *As $G_{\mathbb{Q}_p}$-representations, both V_1 and V_2 are crystalline.*

Proof. The functor $\mathbb{D}_{\mathrm{cris}}$ is compatible with taking direct sums, so we can identify $\mathbb{D}_{\mathrm{cris}}(V_i)$ as a filtered sub-φ-module of $\mathbb{D}_{\mathrm{cris}}(V_f)$ for $i = 1, 2$. That is,

$$\mathbb{D}_{\mathrm{cris}}(\mathrm{Sym}^2(V_f)) \cong \mathbb{D}_{\mathrm{cris}}(V_1) \oplus \mathbb{D}_{\mathrm{cris}}(V_2). \tag{6}$$

Since $\mathrm{Sym}^2(V_f)$ is crystalline, so $\mathbb{D}_{\mathrm{cris}}(\mathrm{Sym}^2(V_f))$ is of dimension 3 over E. Hence, $\mathbb{D}_{\mathrm{cris}}(V_i)$ must have dimension i and V_i is crystalline for $i = 1, 2$. □

We now give explicit descriptions of $\mathbb{D}_{\mathrm{cris}}(V_1)$ and $\mathbb{D}_{\mathrm{cris}}(V_2)$.

Recall that $\mathbb{D}_{\mathrm{cris}}(V_f)$ is a 2-dimensional E-vector space with Hodge-Tate weights 0 and $1-k$. Moreover, the de Rham filtration is given by

$$\mathrm{Fil}^i\,\mathbb{D}_{\mathrm{cris}}(V_f) = \begin{cases} E\omega \oplus E\varphi(\omega) & \text{if } i \leq 0 \\ E\omega & \text{if } 1 \leq i \leq k-1 \\ 0 & \text{if } i \geq k \end{cases} \tag{7}$$

for some $\omega \neq 0$. The action of φ on $\mathbb{D}_{\mathrm{cris}}(V_f)$ satisfies $\varphi^2 = -\epsilon(p)p^{k-1}$. Therefore,

$$\mathrm{Fil}^i\,\mathbb{D}_{\mathrm{cris}}(\mathrm{Sym}^2(V_f)) = \begin{cases} \mathbb{D}_{\mathrm{cris}}(\mathrm{Sym}^2(V_f)) & \text{if } i \leq 0 \\ E(\omega\otimes\omega) \oplus E(\varphi(\omega)\otimes\omega + \omega\otimes\varphi(\omega)) & \text{if } 1 \leq i \leq k-1 \\ E(\omega\otimes\omega) & \text{if } k \leq i \leq 2k-2 \\ 0 & \text{if } i \geq 2k-1 \end{cases} \tag{8}$$

Since $\varphi^2(\omega) = -\epsilon(p)p^{k-1}\omega$, we have

$$\varphi\Big(\omega\otimes\varphi(\omega) + \varphi(\omega)\otimes\omega\Big) = -\epsilon(p)p^{k-1}\Big(\omega\otimes\varphi(\omega) + \varphi(\omega)\otimes\omega\Big).$$

In particular, $\omega\otimes\varphi(\omega) + \varphi(\omega)\otimes\omega$ is an eigenvector of φ. Therefore, we have a decomposition of filtered φ-modules

$$\mathbb{D}_{\mathrm{cris}}(\mathrm{Sym}^2(V_f)) = \Big(E(\omega\otimes\omega) \oplus E(\varphi(\omega)\otimes\varphi(\omega))\Big) \oplus \Big(E(\omega\otimes\varphi(\omega) + \varphi(\omega)\otimes\omega)\Big).$$

Proposition 3.6. *As filtered φ-modules, we have*

$$\begin{aligned} \mathbb{D}_{\mathrm{cris}}(V_1) &= E(\varphi(\omega)\otimes\omega + \omega\otimes\varphi(\omega)), \\ \mathbb{D}_{\mathrm{cris}}(V_2) &= E(\omega\otimes\omega) \oplus E(\varphi(\omega)\otimes\varphi(\omega)). \end{aligned}$$

Proof. By (4), $\rho_1 = \varepsilon_K \cdot \epsilon \cdot \chi^{k-1}$. Since p is inert in K, $\varepsilon_K(p) = -1$. The Hodge-Tate weight of V_1 is therefore $1-k$ and φ acts on $\mathbb{D}_{\mathrm{cris}}(V_1)$ as multiplication by $-\epsilon(p)p^{k-1}$. This proves the first equality. The second equality is then automatic by (6). □

Remark 3.7. *Such a decomposition of $G_{\mathbb{Q}_p}$-representations is in fact possible for f without CM (see* [10, §2.2]*).*

Corollary 3.8. *The eigenvalues of φ on $\mathbb{D}_{\mathrm{cris}}(V_2)$ are $\pm\epsilon(p)p^{k-1}$.*

Proof. By Proposition 3.6, the matrix of φ with respect to the basis $\omega\otimes\omega$, $\varphi(\omega)\otimes\varphi(\omega)$ is

$$\begin{pmatrix} 0 & \epsilon(p)^2p^{2k-2} \\ 1 & 0 \end{pmatrix},$$

hence the result. □

Corollary 3.9. *The Hodge-Tate weights of V_2 are 0 and $2-2k$.*

Proof. This follows from (8) and Proposition 3.6. □

3.4. **The Perrin-Riou pairing.** By Corollary 3.8, the slope of φ on $\mathbb{D}_{\mathrm{cris}}(V_2)$ is $k-1$. Hence, by Corollary 3.9, given any $v \in \mathbb{D}_{\mathrm{cris}}(V_2)$, we have the Perrin-Riou pairing

$$\mathcal{L}_v : \mathbb{H}^1_{\mathrm{Iw}}(V_2^*) \to \mathcal{H}_{k-1,E}(G_\infty)$$

which satisfies the following properties.

Proposition 3.10. *For an integer $r \geq 0$, we have*

$$\chi^r\left(\mathcal{L}_v(\mathbf{z})\right) = r!\left[\left(1 - \frac{\varphi^{-1}}{p}\right)(1-\varphi)^{-1}(v_{r+1}), \exp^*(z_{-r,0})\right]_0.$$

Let θ be a character of G_n which does not factor through G_{n-1} with $n \geq 1$, then

$$\chi^r\theta\left(\mathcal{L}_v(\mathbf{z})\right) = \frac{r!}{\tau(\theta^{-1})}\sum_{\sigma\in G_n}\theta^{-1}(\sigma)\left[\varphi^{-n}(v_{r+1}), \exp^*(z^{\sigma}_{-r,n})\right]_n$$

where $[,]_n$ is the pairing

$$[,]_n : H^1(\mathbb{Q}_{p,n}, V_2(r+1)) \times H^1(\mathbb{Q}_{p,n}, V_2^*(-r)) \to H^2(\mathbb{Q}_{p,n}, E(1)) \cong E,$$

$z_{-r,n}$ denotes the projection of $\mathrm{Tw}_{-r}(z)$ into $H^1(\mathbb{Q}_{p,n}, V_2^*(-r))$ and $\tau(\theta^{-1})$ denotes the Gauss sum of θ^{-1}.

Proof. See [6, §3.2]. □

Remark 3.11. *The assumption on the eigenvalues of φ made in [6] are not necessary for our purposes here because the Perrin-Riou pairings can be defined by applying $1-\varphi$ to the (φ, G_∞)-module of V_2^* (see [7] and [5, §16.4]).*

We fix a non-zero element $\bar{\omega} \in \mathrm{Fil}^{-1}\,\mathbb{D}_{\mathrm{cris}}(V_2^*(1))$ and write

$$\mathrm{per}(\bar{\omega}) = \Omega_+(\gamma')^+ + \Omega_-(\gamma')^-$$

where $\Omega_\pm \in \mathbb{C}^\times$ and γ' is as in the statement of Proposition 3.1 for some fixed γ.

Definition 3.12. *Under the choices made above, we define $v^\pm \in \mathbb{D}_{\mathrm{cris}}(V_2)$ by*

$$v^\pm = \frac{1}{[\varphi(\omega)\otimes\varphi(\omega), \bar{\omega}]}\Big(\pm\epsilon(p)p^{k-1}\omega\otimes\omega + \varphi(\omega)\otimes\varphi(\omega)\Big).$$

Lemma 3.13. *The elements $v^\pm$ satisfy:*

(a) *Both $v^\pm$ are eigenvalues of φ with $\varphi(v^\pm) = \pm\epsilon(p)p^{k-1}v^\pm$;*
(b) *For any $x \in \mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V_2^*(-r))$ and an integer r such that $0 \leq r \leq 2k-3$, we have*

$$[v^+_{r+1}, x] = [v^-_{r+1}, x]$$

where $[,]$ denotes the pairing

$$[,] : \mathbb{D}_{\mathrm{cris}}(V_2(r+1)) \times \mathbb{D}_{\mathrm{cris}}(V_2^*(-r)) \to \mathbb{D}_{\mathrm{cris}}(E(1)) = E\cdot t^{-1}e_1.$$

Proof. (a) is easy to check using the matrix given in the proof of Corollary 3.8 (or by direct calculations).

By Corollary 3.9, the Hodge-Tate weights of V_2^* are 0 and $2k-2$. Hence, $\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V_2^*(-r))$ is one-dimensional with basis $\bar{\omega}_{-r-1}$ for $0 \leq r \leq 2k-3$. Since $(\omega\otimes\omega)_{r+1} \in \mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V_2(r+1))$, we have $[(\omega\otimes\omega)_{r+1}, \bar{\omega}_{-r-1}] = 0$. Hence,

$$[v^+_{r+1}, \bar{\omega}_{-r-1}] = [v^-_{r+1}, \bar{\omega}_{-r-1}] = 1,$$

which implies (b). □

Note that $V_2^* \cong \tilde{V}_{\bar{\phi}^2}(2k-2)$. This enables us to make the following definition of p-adic L-functions associated to ϕ^2.

Definition 3.14. *On taking $\eta = \bar{\phi}^2$ in §3.1, we define*

$$L_{\pm\epsilon(p)p^{k-1}}(\phi^2) = \mathcal{L}_{v^\pm}\left(\mathrm{Tw}_{2k-2}\left(z(\bar{\phi}^2)\right)\right) \in \mathcal{H}_{k-1,E}(G_\infty).$$

Lemma 3.15. *Let θ be a character of G_n which does not factor through G_{n-1} with $n \geq 1$ and write $\delta = \theta(-1)$, then*

$$\chi^{2k-3}\theta\Big(L_\alpha(\phi^2)\Big) = \frac{(2k-3)!p^{(2k-2)n}}{\tau(\theta^{-1})\alpha^n} \times \frac{L(\phi^2\theta^{-1}, 2k-2)}{\Omega_\delta}$$

where $\alpha = \pm\epsilon(p)p^{k-1}$.

Proof. We have

$$\begin{aligned}
& \chi^{2k-3}\theta\Big(L_{\pm\epsilon(p)p^{k-1}}(\phi^2)\Big)\\
= & \chi^{2k-3}\theta\Big(\mathcal{L}_{v^\pm}\left(\mathrm{Tw}_{2k-2}\left(z(\bar{\phi}^2)\right)\right)\Big)\\
= & \frac{(2k-3)!}{\tau(\theta^{-1})}\sum_{\sigma\in G_n}\theta^{-1}(\sigma)\left[\varphi^{-n}(v_{2k-2}^{\pm}),\exp^*(z_{1,n}(\bar{\phi}^2)^\sigma)\right]_n\\
= & \frac{(2k-3)!}{\tau(\theta^{-1})}\left[\left(\pm\epsilon(p)p^{k-1}\times p^{-2k+2}\right)^{-n}v_{2k-2}^{\pm},\sum_{\sigma\in G_n}\theta^{-1}(\sigma)\exp^*(z_{1,n}(\bar{\phi}^2)^\sigma)\right]_n\\
= & \frac{(2k-3)!p^{(2k-2)n}}{\tau(\theta^{-1})\left(\pm\epsilon(p)p^{k-1}\right)^n}\times\frac{L(\phi^2\theta^{-1},2k-2)}{\Omega_\delta}
\end{aligned}$$

where the second equality follows from Proposition 3.10, the third follows from Lemma 3.13(a) and the last equality is a consequence of Proposition 3.1 and the fact that p divides the conductor of θ. $\square$

Lemma 3.16. *We have*

$$\chi^{2k-3}\Big(L_{\pm\epsilon(p)p^{k-1}}(\phi^2)\Big)=\Big(1-p^{-1}+(1-\epsilon(p)^{-2}p^{2k-3})(\pm\epsilon(p)p^{1-k})\Big)\times\frac{L(\phi^2,2k-2)}{\Omega_+}.$$

Proof. Since $\varphi^2=\epsilon(p)^2p^{2-2k}$ on $\mathbb{D}_{\mathrm{cris}}(V_2(2k-2))$, we have

$$\begin{aligned}
& \left(1-\frac{\varphi^{-1}}{p}\right)(1-\varphi)^{-1}\\
= & (1-\epsilon(p)^{-2}p^{2k-3}\varphi)\frac{1+\varphi}{1-\epsilon(p)^2p^{2-2k}}\\
= & \frac{1-p^{-1}+(1-\epsilon(p)^{-2}p^{2k-3})\varphi}{1-\epsilon(p)^2p^{2-2k}}.
\end{aligned}$$

Therefore, similarly to the proof of Lemma 3.15, we have

$$\begin{aligned}
& \chi^{2k-3}\Big(L_{\pm\epsilon(p)p^{k-1}}(\phi^2)\Big)\\
= & \chi^{2k-3}\Big(\mathcal{L}_{v^\pm}\left(\mathrm{Tw}_{2k-2}\left(z(\bar{\phi}^2)\right)\right)\Big)\\
= & (2k-3)!\left[\frac{1-p^{-1}+(1-\epsilon(p)^{-2}p^{2k-3})\varphi}{1-\epsilon(p)^2p^{2-2k}}\left(v_{2k-2}^{\pm}\right),\exp^*(z_{1,0}(\bar{\phi}^2))\right]_0\\
= & (2k-3)!\left[\frac{1-p^{-1}+(1-\epsilon(p)^{-2}p^{2k-3})(\pm\epsilon(p)p^{1-k})}{1-\epsilon(p)^2p^{2-2k}}\cdot v_{2k-2}^{\pm},\exp^*(z_{1,0}(\bar{\phi}^2))\right]_0\\
= & \frac{1-p^{-1}+(1-\epsilon(p)^{-2}p^{2k-3})(\pm\epsilon(p)p^{1-k})}{1-\epsilon(p)^2p^{2-2k}}\times\frac{L_{\{p\}}(\phi^2,2k-2)}{\Omega_+}\\
= & \Big(1-p^{-1}+(1-\epsilon(p)^{-2}p^{2k-3})(\pm\epsilon(p)p^{1-k})\Big)\times\frac{L(\phi^2,2k-2)}{\Omega_+}.
\end{aligned}$$

$\square$

Remark 3.17. *Consider the p-adic L-function $L_{+\epsilon(p)p^{k-1}}(\phi^2)$. The first factor on the right-hand side of the equation in the statement of Lemma 3.16 vanishes if and only if $k=2$ and $\epsilon(p)=1$ (e.g. when f corresponds to an elliptic curve over $\mathbb{Q}$). This recovers the trivial zero result in* [10].

3.5. **p-adic L-functions of the symmetric square.** Let us first recall the following result of Kubota and Leopoldt.

Theorem 3.18. *If η is a non-trivial Dirichlet character of conductor prime to p, there exists a bounded p-adic measure $L_p(\eta) \in \mathcal{H}_{0,E}(G_\infty)$ where E is some finite extension of $\mathbb{Q}_p$ which contains the image of η such that*

$$\begin{aligned} \chi^r\theta(L_p(\eta)) &= \frac{(r+1)!p^{n(r+1)}}{(2\pi i)^{r+1}\tau(\theta^{-1})} \times L(\eta\theta^{-1}, r+1); \\ \chi^r(L_p(\eta)) &= \frac{(r+1)!}{(2\pi i)^{r+1}} L(\eta, r+1). \end{aligned}$$

for any integer $r \geq 0$ and Dirichlet character θ of conductor p^n such that $\chi^{r+1}\theta(-1) = \eta(-1)$.

Since we assume that Hypothesis 2.1 holds, we may take $\eta = \varepsilon_K \cdot \epsilon$ in Theorem 3.18. This enables us to give the following definition.

Definition 3.19. *For $\alpha = \pm\epsilon(p)p^{k-1}$ we define*

$$L_\alpha\left(\mathrm{Sym}^2(V_f)\right) = L_\alpha(\phi^2) \times \mathrm{Tw}_{-k+1}\left(L_p(\varepsilon_K \cdot \epsilon)\right).$$

For the rest of this section, unless otherwise stated, θ denotes an even character on G_n which does not factor through G_{n-1} with $n \geq 1$.

Theorem 3.20. *Both $L_{\pm\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right)$ lie inside $\mathcal{H}_{k-1,E}(G_\infty)$ and admit the following interpolating properties:*

$$\begin{aligned} \chi^{2k-3}\theta\Big(L_\alpha\left(\mathrm{Sym}^2(V_f)\right)\Big) &= \frac{(2k-3)!(k-1)!p^{3n(k-1)}}{\tau(\theta^{-1})^2\alpha^n} \times \frac{L(\mathrm{Sym}^2 f, \theta^{-1}, 2k-2)}{(2\pi i)^{k-1}\Omega_+}; \\ \chi^{2k-3}\Big(L_\alpha\left(\mathrm{Sym}^2(V_f)\right)\Big) &= (2k-3)!(k-1)!\left(1 - \frac{1}{p} + \alpha\left(p^{-2k+2} - \frac{1}{p\epsilon(p)^2}\right)\right) \times \frac{L(\mathrm{Sym}^2 f, 2k-2)}{(2\pi i)^{k-1}\Omega_+} \end{aligned}$$

where $\alpha = \pm\epsilon(p)p^{k-1}$.

Proof. By definition, $L_\alpha(\phi^2) \in \mathcal{H}_{k-1,E}(G_\infty)$ and $L_p(\varepsilon_K \cdot \epsilon) \in \mathcal{H}_{0,E}(G_\infty)$ which implies the first part of the theorem.

Since $\det(V_f) = \epsilon\chi^{k-1}$ and ρ_f is odd, we have $\epsilon\chi^{k-1}(-1) = -1$. But $\varepsilon_K(-1) = -1$ and $\theta(-1) = 1$, so $\chi^{k-1}\theta(-1) = \varepsilon_K\epsilon(-1)$ and we can apply Theorem 3.18 and Lemma 3.15 as follows:

$$\begin{aligned} &\chi^{2k-3}\theta\Big(L_\alpha\left(\mathrm{Sym}^2(V_f)\right)\Big) \\ = \;& \chi^{2k-3}\theta\Big(L_\alpha(\phi^2)\Big) \times \chi^{k-2}\theta\Big(L_p(\varepsilon_K \cdot \epsilon)\Big) \\ = \;& \frac{(2k-3)!p^{(2k-2)n}}{\tau(\theta^{-1})\alpha^n} \times \frac{L(\phi^2\theta, 2k-2)}{\Omega_+} \times \frac{(k-1)!p^{n(k-1)}}{(2\pi i)^{k-1}\tau(\theta^{-1})} \times L(\varepsilon_k \cdot \epsilon \cdot \theta^{-1}, k-1) \\ = \;& \frac{(2k-3)!(k-1)!p^{3n(k-1)}}{\tau(\theta^{-1})^2\alpha^n} \times \frac{L(\mathrm{Sym}^2 f, \theta^{-1}, 2k-2)}{(2\pi i)^{k-1}\Omega_+}, \end{aligned}$$

where the last equality follows from Corollary 3.4. This gives the first interpolating formula and the second one can be deduced in the same way. ☐

Lemma 3.21. *Let η be an even character on Δ, then $L^\eta_{\pm\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right) \neq 0$.*

Proof. We have $L(\mathrm{Sym}^2(V_f), \eta, 2k-2) \neq 0$ because the critical strip of $\mathrm{Sym}^2(V_f)$ is $k-1 < \mathrm{Re}(s) < k$. Therefore, we are done by the interpolating properties given by Theorem 3.20. ☐

3.6. **Pollack's plus and minus splittings.** As in [11], we define

$$\log^+(\gamma) = \prod_{r=0}^{2k-3}\prod_{n=1}^{\infty}\frac{\Phi_{2n}(\chi(\gamma)^{-r}\gamma)}{p},$$
$$\log^-(\gamma) = \prod_{r=0}^{2k-3}\prod_{n=1}^{\infty}\frac{\Phi_{2n-1}(\chi(\gamma)^{-r}\gamma)}{p},$$

where Φ_m denotes the p^mth cyclotomic polynomial. Then, $\log^{\pm}(\gamma)\sim\log^{k-1}$.

Lemma 3.22. *For an integer r such that $0\le r\le 2k-3$ and a character θ of G_n which does not factor through G_{n-1} with $n\ge 1$, then*

$$\chi^r\theta\Big(L_{+\epsilon(p)p^{k-1}}(\phi^2)\Big)=(-1)^n\chi^r\theta\Big(L_{-\epsilon(p)p^{k-1}}(\phi^2)\Big).$$

Proof. This follows from the same calculations as in the proof of Lemma 3.15 thanks to Lemma 3.13(b). □

Corollary 3.23. *We have divisibilities*

$$\log^+(\gamma) \mid L_{+\epsilon(p)p^{k-1}}(\phi^2)+L_{-\epsilon(p)p^{k-1}}(\phi^2);$$
$$\log^-(\gamma) \mid L_{+\epsilon(p)p^{k-1}}(\phi^2)-L_{-\epsilon(p)p^{k-1}}(\phi^2).$$

Similarly,

$$\log^+(\gamma) \mid L_{+\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right)+L_{-\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right);$$
$$\log^-(\gamma) \mid L_{+\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right)-L_{-\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right).$$

Proof. The first set of divisibilities follows from Lemma 3.22. The second set is then immediate by definition. □

This allows us to define the following.

Definition 3.24. *We define the plus and minus p-adic L-functions for $\mathrm{Sym}^2(V_f)$ by*

$$L_p^+(\mathrm{Sym}^2(V_f)) = \Big(L_{+\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right)+L_{-\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right)\Big)/2\log^+(\gamma);$$
$$L_p^-(\mathrm{Sym}^2(V_f)) = \Big(L_{+\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right)-L_{-\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right)\Big)/2\log^-(\gamma).$$

Similarly, we define the plus and minus p-adic L-functions for V_2 by

$$L_p^+(\phi^2) = \Big(L_{+\epsilon(p)p^{k-1}}\left(\phi^2\right)+L_{-\epsilon(p)p^{k-1}}\left(\phi^2\right)\Big)/2\log^+(\gamma);$$
$$L_p^-(\phi^2) = \Big(L_{+\epsilon(p)p^{k-1}}\left(\phi^2\right)-L_{-\epsilon(p)p^{k-1}}\left(\phi^2\right)\Big)/2\log^-(\gamma).$$

It is immediate that we have

$$L_p^{\pm}\left(\mathrm{Sym}^2(V_f)\right)=L_p^{\pm}(\phi^2)\times\mathrm{Tw}_{-k+1}\left(L_p(\varepsilon_K\cdot\epsilon)\right). \tag{9}$$

Theorem 3.25. *Both $L_p^{\pm}(\mathrm{Sym}^2(V_f))$ are elements of $\Lambda_E(G_\infty)$ and admit the following interpolating properties:*

(a) *If n is even, then*

$$\chi^{2k-3}\theta\Big(L_p^+(\mathrm{Sym}^2(V_f))\Big) = \frac{(2k-3)!(k-1)!p^{2n(k-1)}}{\log^+\left(\chi^{2k-3}\theta(\gamma)\right)\tau(\theta^{-1})^2\epsilon(p)^n}\times\frac{L(\mathrm{Sym}^2 f,\theta^{-1},2k-2)}{(2\pi i)^{k-1}\Omega_+},$$
$$\chi^{2k-3}\Big(L_p^+(\mathrm{Sym}^2(V_f))\Big) = \frac{(2k-3)!(k-1)!\left(1-p^{-1}\right)}{\log^+\left(\chi^{2k-3}(\gamma)\right)}\times\frac{L(\mathrm{Sym}^2 f,2k-2)}{(2\pi i)^{k-1}\Omega_+};$$

(b) *If n is odd, then*

$$\begin{aligned}
\chi^{2k-3}\theta\Big(L_p^-(\mathrm{Sym}^2(V_f))\Big) &= \frac{(2k-3)!(k-1)!p^{2n(k-1)}}{\log^-\left(\chi^{2k-3}\theta(\gamma)\right)\tau(\theta^{-1})^2\epsilon(p)^n}\times\frac{L(\mathrm{Sym}^2 f,\theta^{-1},2k-2)}{(2\pi i)^{k-1}\Omega_+},\\
\chi^{2k-3}\Big(L_p^-(\mathrm{Sym}^2(V_f))\Big) &= \frac{(2k-3)!(k-1)!\left(\epsilon(p)p^{-k+1}-\epsilon(p)^{-1}p^{k-2}\right)}{\log^-\left(\chi^{2k-3}(\gamma)\right)}\times\frac{L(\mathrm{Sym}^2 f,2k-2)}{(2\pi i)^{k-1}\Omega_+}.
\end{aligned}$$

Moreover, $L_p^{\pm}(\mathrm{Sym}^2(V_f))$ are uniquely determined by (a) and (b) respectively.

Proof. By the first part of Theorem 3.20, $L_{\pm\epsilon(p)p^{k-1}}\left(\mathrm{Sym}^2(V_f)\right)$ are both elements of $\mathcal{H}_{k-1,E}(G_\infty)$. But $\log^{\pm}(\gamma)\sim\log^{k-1}$, so the quotients above are in $\mathcal{H}_{0,E}(G_\infty)=\Lambda_E(G_\infty)$.

The interpolating formulae in (a) and (b) follow from those given in Theorem 3.20.

Finally, since $L_p^{\pm}(\mathrm{Sym}^2(V_f))\in\Lambda_E(G_\infty)$, they are uniquely determined by their values at an infinite number of characters, hence the last part of the theorem. □

Lemma 3.26. *Let η be an even character on Δ, then $L_p^{\pm,\eta}\left(\mathrm{Sym}^2(V_f)\right)\neq 0$.*

Proof. The same as the proof of Lemma 3.21. □

Remark 3.27. *Analogues of Theorem 3.25 and Lemma 3.26 for $L_p^{\pm}(\phi^2)$ can be deduced in the same way.*

Remark 3.28. *A conjectural generalisation of Pollack's plus and minus splittings of p-adic L-functions for motives has been formulated in* [2]*. Theorem 3.25 gives an affirmative answer to Conjecture 2 of op. cit. for the special case when the motive corresponds to the symmetric square of a CM modular form.*

4. Selmer groups

In this section, we define the plus and minus p-Selmer groups for $\mathrm{Sym}^2(V_f)$ and relate them to the p-adic L-functions $L_p^{\pm}\left(\mathrm{Sym}^2(V_f)\right)$ defined above. By the decomposition given by Proposition 3.3, we only need to define their counterparts for $V_2=\tilde{V}_{\phi^2}$ because the Selmer group of V_1 is relatively well-understood. The $G_{\mathbb{Q}}$-representation V_2 behaves in exactly the same way as $V_{f'}$ where f' is some CM modular form of weight $2k-1$, so many of the results on V_2 below can be proved using the arguments given in [6]. Therefore, we only outline the proofs without giving all the details here.

4.1. Coleman maps and Selmer groups.

As in [6, 7], we define plus and minus Selmer groups using the kernels of some Coleman maps.

Proposition 4.1. *If $z\in\mathbb{H}^1_{\mathrm{Iw}}(V_2^*)$, then*

$$\begin{aligned}
\log^+(\gamma) &\mid \mathcal{L}_{\varphi(\omega)\otimes\varphi(\omega)}(z),\\
\log^-(\gamma) &\mid \mathcal{L}_{\omega\otimes\omega}(z).
\end{aligned}$$

Proof. As in [6, Proposition 3.14], this can be proved using Proposition 3.10. □

Therefore, as in [6], we may define $\Lambda_E(G_\infty)$-homomorphisms

$$\begin{aligned}
\mathrm{Col}^+:\mathbb{H}^1_{\mathrm{Iw}}(V_2^*) &\to \Lambda_E(G_\infty)\\
z &\mapsto \frac{1}{2[\varphi(\omega)\otimes\varphi(\omega),\bar{\omega}]\log^+(\gamma)}\mathcal{L}_{\varphi(\omega)\otimes\varphi(\omega)}(z);\\
\mathrm{Col}^-:\mathbb{H}^1_{\mathrm{Iw}}(V_2^*) &\to \Lambda_E(G_\infty)\\
z &\mapsto \frac{1}{2[\varphi(\omega)\otimes\varphi(\omega),\bar{\omega}]\log^-(\gamma)}\mathcal{L}_{\omega\otimes\omega}(z).
\end{aligned}$$

Then, it is clear by definition that $\mathrm{Col}^{\pm}\left(\mathrm{Tw}_{2k-2}\left(z(\bar{\phi}^2)\right)\right)=L_p^{\pm}(\phi^2)$.

We now fix an $\mathcal{O}_E$-lattice T of $V(\phi)$ which is stable under $G_{\mathbb{Q}}$, it then gives rise to natural $\mathcal{O}_E$-lattices $T_f = \mathrm{Ind}_K^{\mathbb{Q}}(T)$ and $\mathrm{Sym}^2 T_f$ in $V_f = \tilde{V}_\phi$ and $\mathrm{Sym}^2(V_f)$ respectively, both of which are again stable under $G_{\mathbb{Q}}$. As $p \neq 2$, we have

$$\mathrm{Sym}^2 T_f \cong T_1 \oplus T_2 \quad \text{and} \quad \mathrm{Sym}^2 V_f/T_f \cong V_1/T_1 \oplus V_2/T_2$$

for some $\mathcal{O}_E$-lattice T_i inside V_i for $i = 1, 2$.

Write $H^1_\pm(\mathbb{Q}_{p,n}, T_2^*)$ for the projection of $\ker(\mathrm{Col}^\pm)$ into $H^1(\mathbb{Q}_{p,n}, T_2^*)$ and define $H^1(\mathbb{Q}_{p,n}, V_2/T_2(1))^\pm$ to be the exact annihilator of $H^1_\pm(\mathbb{Q}_{p,n}, T_2^*)$ under the Pontryagin duality

$$H^1(\mathbb{Q}_{p,n}, T_2^*) \times H^1(\mathbb{Q}_{p,n}, V_2/T_2(1)) \to \mathbb{Q}_p/\mathbb{Z}_p.$$

Let F be a number field. Then the p-Selmer group of $\mathrm{Sym}^2 T_f(1)$ decomposes into those of $T_1(1)$ and $T_2(1)$:

$$\mathrm{Sel}_p(\mathrm{Sym}^2 T_f(1)/F) = \mathrm{Sel}_p(T_1(1)/F) \oplus \mathrm{Sel}_p(T_2(1)/F).$$

We define the plus/minus Selmer groups over $k_n = \mathbb{Q}(\mu_{p^n})$ by

$$\begin{aligned} \mathrm{Sel}_p^\pm(T_2(1)/k_n) &= \ker\left(\mathrm{Sel}_p(T_2(1)/k_n) \to \frac{H^1(\mathbb{Q}_{p,n}, V_2/T_2(1))}{H^1_f(\mathbb{Q}_{p,n}, V_2/T_2(1))^\pm}\right), \\ \mathrm{Sel}_p^\pm(\mathrm{Sym}^2 T_f(1)/k_n) &= \mathrm{Sel}_p(T_1(1)/k_n) \oplus \mathrm{Sel}_p^\pm(T_2(1)/k_n) \end{aligned}$$

and let

$$\mathrm{Sel}_p^\pm(T_2(1)/k_\infty) = \varinjlim \mathrm{Sel}_p^\pm(T_2(1)/k_n) \quad \text{and} \quad \mathrm{Sel}_p^\pm(\mathrm{Sym}^2 T_f(1)/k_\infty) = \varinjlim \mathrm{Sel}_p^\pm(\mathrm{Sym}^2 T_f(1)/k_n).$$

4.2. **Description of the kernels.** In this section, we give a more explicit description of the groups $H^1_f(\mathbb{Q}_{p,n}, V_2/T_2(1))^\pm$ under the following additional assumption.

Hypothesis 4.2. *Either $p-1 \nmid k-1$ or $\epsilon \neq 1$.*

In [6, §4], one of the key ingredients to give an explicit description of $H^1_f(\mathbb{Q}_{p,n}, V_f/T_f(1))^\pm$ is the fact that $(V_f/T_f(j))^{G_{\mathbb{Q}_{p,n}}} = 0$ under some appropriate assumptions. We show below that we get an analogue of such description under Hypothesis 4.2.

Lemma 4.3. *If Hypothesis 4.2 holds, then $(V_2/T_2(j))^{G_{\mathbb{Q}_{p,n}}} = 0$ for all $j \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Let $q \nmid N$ be a prime which is inert in K. Then, by the second half of the proof of Proposition 3.3, we see that the eigenvalues of the q-Frobenius on $V_2(j)$ are $\pm\epsilon(q)\chi^j(q)q^{k-1}$. Therefore, as in [6, proof of Lemma 4.4], it is enough to show that there exists some q such that

$$\pm\epsilon(q)\chi^j(q)q^{k-1} \not\equiv 1 \mod p.$$

If either $p-1 \nmid k-1$ or $\epsilon(q) \neq 1$, we can find such a q by Dirichlet's theorem, so we are done. $\square$

Corollary 4.4. *If Hypothesis 4.2 holds, then the restriction map $H^1(\mathbb{Q}_{p,m}, T_2(1)) \to H^1(\mathbb{Q}_{p,n}, T_2(1))$ is injective for any integers $n \geq m \geq 0$. On identifying the former as a subgroup of the latter, we have*

$$H^1_f(\mathbb{Q}_{p,n}, V_2/T_2(1))^\pm = H^1_f(\mathbb{Q}_{p,n}, T_2(1))^\pm \otimes E/\mathcal{O}_E.$$

Here

$$H^1_f(\mathbb{Q}_{p,n}, T_2(1))^\pm = \left\{x \in H^1_f(\mathbb{Q}_{p,n}, T_2(1)) : \mathrm{cor}_{n/m+1}(x) \in H^1_f(\mathbb{Q}_{p,m}, T_2(1)) \forall m \in S_n^\pm\right\}$$

where cor *denotes the corestriction map and*

$$\begin{aligned} S_n^+ &= \{m \in [0, m-1] : m \textit{ even}\}, \\ S_n^- &= \{m \in [0, m-1] : m \textit{ odd}\}. \end{aligned}$$

Proof. These can be proved in exactly the same way as their counterparts in [6, §4] using Lemma 4.3. $\square$

4.3. **Main conjectures.**

Theorem 4.5. *Let θ be a character on Δ and $r \geq 0$ an integer such that $\chi^{r+1}\theta(-1) = \eta(-1)$. Then* $\mathrm{Sel}_p\left(\mathbb{Z}_p(\eta)(r+1)\right)^{\theta}$ *is $\Lambda_E(\Gamma)$-cotorsion and*

$$\mathrm{Char}_{\Lambda_E(\Gamma)}\left(\mathrm{Sel}_p\left(\mathbb{Z}_p(\eta)(r+1)\right)^{\vee,\theta}\right) = \left(\mathrm{Tw}_{-r}\, L_p^{\theta}(\eta)\right).$$

Proof. For any $\Lambda_E(G_\infty)$-module, $M^\vee(r) = M(-r)^\vee$. If M is a $\Lambda_E(\Gamma)$-torsion module, we have $\mathrm{Char}(M(r)) = \mathrm{Tw}_r(\mathrm{Char}(M))$. Therefore, the result is just a rewrite of the Iwasawa main conjecture, as proved by Mazur-Wiles [8]. □

Corollary 4.6. *Let η be an even character on Δ. Then*

$$\mathrm{Char}_{\Lambda_E(\Gamma)}\left(\mathrm{Sel}_p(T_1(1)/k_\infty)^{\vee,\eta}\right) = \left(\mathrm{Tw}_{-k+1}\, L_p^{\eta}(\varepsilon_K \cdot \epsilon)\right).$$

Proof. We may apply Theorem 4.5 to $\varepsilon_K \cdot \epsilon$ with $r = k-1$. □

Proposition 4.7. *Let $\delta = \pm$ and let η be a character on Δ such that $\eta = 1$ if $\delta = -$. Then,* $\mathrm{Sel}_p^{\delta}(T_2(1)/k_\infty)^{\theta}$ *is $\Lambda_E(\Gamma)$-cotorsion and*

$$\mathrm{Char}_{\Lambda_E(\Gamma)}\left(\mathrm{Sel}_p^{\delta}(T_2(1)/k_\infty)^{\vee,\eta}\right) = \left(L_p^{\delta,\eta}(\phi^2)\right).$$

Proof. This follows from the same argument as in [12], which has been generalised for CM modular forms in [6, §7]. It relies on the main conjecture for K as proved in [13]. □

Theorem 4.8. *Let η be character on Δ as in the statement of Proposition 4.7. Then* $\mathrm{Sel}_p^{\pm}(\mathrm{Sym}^2(V_f)/k_\infty)^{\eta}$ *is $\Lambda_E(\Gamma)$-cotorsion and*

$$\mathrm{Char}_{\Lambda_E(\Gamma)}\left(\mathrm{Sel}_p^{\pm}(\mathrm{Sym}^2(V_f)/k_\infty)^{\vee,\eta}\right) = \left(L_p^{\pm,\eta}(\mathrm{Sym}^2(V_f))\right).$$

Proof. Recall that

$$\mathrm{Sel}_p^{\pm}(\mathrm{Sym}^2 T_f(1)/k_\infty) = \mathrm{Sel}_p(T_1(1)/k_\infty) \oplus \mathrm{Sel}_p^{\pm}(T_2(1)/k_\infty)$$

by definition, so

$$\mathrm{Sel}_p^{\pm}(\mathrm{Sym}^2 T_f(1)/k_\infty)^{\vee,\eta} = \mathrm{Sel}_p(T_1(1)/k_\infty)^{\vee,\eta} \oplus \mathrm{Sel}_p^{\pm}(T_2(1)/k_\infty)^{\vee,\eta}.$$

But we have

$$L_p^{\pm,\eta}\left(\mathrm{Sym}^2(V_f)\right) = L_p^{\pm,\eta}(\phi^2) \times \mathrm{Tw}_{-k+1}\left(L_p^{\eta}(\varepsilon_K \cdot \epsilon)\right)$$

by (9). Therefore, the theorem follows from Corollary 4.6 and Proposition 4.7 because

$$\mathrm{Char}(M_1 \oplus M_2) = \mathrm{Char}(M_1)\,\mathrm{Char}(M_2)$$

for any torsion modules M_1 and M_2. □

5. Appendix

In this section, we fix an integer $m \geq 2$. We prove an analogue of Proposition 3.3.

Proposition 5.1. *If m is even, we have a decomposition of $G_{\mathbb{Q}}$-representations*

$$\mathrm{Sym}^m V_f \cong \bigoplus_{i=0}^{m/2-1}\left(\tilde{V}_{\phi^{m-2i}} \otimes (\varepsilon_K \det\rho_f)^i\right) \oplus (\varepsilon_K \det\rho_f)^{m/2}.$$

If m is odd, then

$$\mathrm{Sym}^m V_f \cong \bigoplus_{i=0}^{(m-1)/2}\left(\tilde{V}_{\phi^{m-2i}} \otimes (\varepsilon_K \det\rho_f)^i\right).$$

Proof. We only give the proof for the case when m is even since the other case can be proved in a similar way. Let x, y be the basis of V_f given as in §3.2. For an integer r such that $0 \le r \le m$, we write x_r for the element in $V_f^{\otimes m}$ given by

$$\sum a_1 \otimes a_2 \otimes \cdots \otimes a_m$$

where the sum runs over $a_i \in \{x, y\}$ with $\#\{i : a_i = x\} = r$. Then, $x_0, \ldots, x_m$ give a basis of $\mathrm{Sym}^m V_f$.

If $\sigma \in G_K$, we have

$$\sigma(x_r) = \tilde{\phi}^r(\sigma)\tilde{\phi}^{m-r}(\iota\sigma\iota)x_r$$

by (2). If $\sigma = \iota\sigma'$ with $\sigma' \in G_K$, then

$$\sigma(x_r) = \tilde{\phi}^r(\sigma')\tilde{\phi}^{m-r}(\iota\sigma'\iota)x_{m-r}$$

by (3). Therefore, x_r and x_{m-r} generate a subrepresentation of $\mathrm{Sym}^m V_f$, which we denote by $\rho_r : G_{\mathbb{Q}} \to \mathrm{GL}(V_r)$ where $0 \le r \le m/2$. Note that V_r is 2-dimensional if $r < m/2$ and $V_{m/2}$ is 1-dimensional. We have a decomposition

$$\mathrm{Sym}^m V_f \cong \oplus_{r=0}^{m/2} V_r.$$

For $r < m/2$, the matrix of $\sigma \in G_K$ respect to the basis x_{m-r}, x_r is

$$\begin{pmatrix} \tilde{\phi}^{m-r}(\sigma)\tilde{\phi}^r(\iota\sigma\iota) & 0 \\ 0 & \tilde{\phi}^r(\sigma)\tilde{\phi}^{m-r}(\iota\sigma\iota) \end{pmatrix} = \tilde{\phi}^r(\sigma\iota\sigma\iota) \begin{pmatrix} \tilde{\phi}^{m-2r}(\sigma) & 0 \\ 0 & \tilde{\phi}^{m-2r}(\iota\sigma\iota) \end{pmatrix},$$

whereas that of $\sigma = \iota\sigma'$ with $\sigma' \in G_K$ is given by

$$\begin{pmatrix} 0 & \tilde{\phi}^r(\sigma')\tilde{\phi}^{m-r}(\iota\sigma'\iota) \\ \tilde{\phi}^{m-r}(\sigma')\tilde{\phi}^r(\iota\sigma'\iota) & 0 \end{pmatrix} = \tilde{\phi}^r(\sigma'\iota\sigma'\iota) \begin{pmatrix} 0 & \tilde{\phi}^{m-2r}(\iota\sigma'\iota) \\ \tilde{\phi}^{m-2r}(\sigma') & 0 \end{pmatrix}.$$

Therefore, we see that $\rho_r \cong \mathrm{Ind}_K^{\mathbb{Q}}(V(\phi^{m-2r})) \cdot (\varepsilon_K \det \rho_f)^r$ by Lemma 3.2.

Finally, for $r = m/2$, we have

$$\sigma(x_{m/2}) = \begin{cases} \tilde{\phi}^{m/2}(\sigma\iota\sigma\iota)x_{m/2} & \text{if } \sigma \in G_K \\ \tilde{\phi}^{m/2}(\sigma'\iota\sigma'\iota)x_{m/2} & \text{if } \sigma = \iota\sigma' \text{ where } \sigma' \in G_K. \end{cases}$$

Hence, $V_{m/2} = (\varepsilon_K \det \rho_f)^{m/2}$ again by Lemma 3.2. This finishes the proof. □

Corollary 5.2. *The complex L-function admits a factorisation*

$$L(\mathrm{Sym}^m f, s) = \begin{cases} \left(\prod_{i=0}^{m/2-1} L\left(\phi^{m-2i}, (\varepsilon_K\epsilon)^i, s - i(k-1)\right)\right) L\left((\varepsilon_K\epsilon)^{m/2}, s - m/2(k-1)\right) & \text{if } m \text{ is even,} \\ \prod_{i=0}^{(m-1)/2} L\left(\phi^{m-2i}, (\varepsilon_K\epsilon)^i, s - i(k-1)\right) & \text{otherwise.} \end{cases}$$

Proof. This can be proved in the same way as Corollary 3.4. □

Remark 5.3. *For $0 \le i \le \lfloor (m-1)/2 \rfloor$, we may obtain a p-adic L-function that interpolates the L-values of ϕ^{m-2i} at $(m-2i)(k-1)$ using Proposition 3.1. However, when $m > 2$, their product does not interpolate the L-values of $\mathrm{Sym}^m f$. We would need p-adic L-functions that interpolate the L-values of ϕ^{m-2i} at $(m-i)(k-1)$ instead.*

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