IWASAWA THEORY FOR THE SYMMETRIC SQUARE OF A CM MODULAR FORM AT INERT PRIMES

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Abstract. Let \( f \) be a CM modular form and \( p \) an odd prime which is inert in the CM field. We construct two \( p \)-adic \( L \)-functions for the symmetric square of \( f \), one of which has the same interpolating properties as the one constructed by Delbourgo-Dabrowski, whereas the second one has a similar interpolating properties but corresponds to a different eigenvalue of the Frobenius. The symmetry between these two \( p \)-adic \( L \)-functions allows us to define the plus and minus \( p \)-adic \( L \)-functions à la Pollack. We also define the plus and minus \( p \)-Selmer groups analogous to Kobayashi’s Selmer groups. We explain how to relate these two sets of objects via a main conjecture.

1. Introduction

Let \( f \) be a normalised eigen-newform of weight \( k \), level \( N \) and character \( \epsilon \). Fix a prime \( p \not\equiv 2 \) such that \( p \nmid N \). In [3] (also in [1] under some additional conditions), even distributions on \( \mathbb{Z}_p^* \) are constructed to interpolate the \( L \)-values of the symmetric square of \( f \). More precisely, if the Euler factor of \( L(E,s) \) at \( p \) is given by \( (1-\alpha_1(p)p^{-s})(1-\alpha_2(p)p^{-s}) \), then there exists an admissible distribution \( \mu_{\epsilon,(p)^i} \) for \( i = 1,2 \) such that

\[
\int_{\mathbb{Z}_p^*} \theta d\mu_{\epsilon,(p)^i} = \frac{p^{3n(k-1)}}{\alpha_i(p)^{2p^n}(\theta^{-1})} \times \frac{L(\text{Sym}^2 f,\theta^{-1},2k-2)}{(\text{period})}
\]

for any non-trivial even Dirichlet character \( \theta \) of conductor \( p^n \) where \( \tau(\theta^{-1}) \) denotes the Gauss sum of \( \theta^{-1} \).

Since the Euler factor of \( L(\text{Sym}^2 f,s) \) at \( p \) is \( (1-\alpha_1(p)^2p^{-s})(1-\alpha_2(p)^2p^{-s})(1-\epsilon(p)p^{k-1-s}) \), we expect that there should be a distribution \( \mu_{\epsilon(p)p^{k-1}} \) satisfying interpolating properties similar to (1), but with \( \alpha_i(p)^2 \) replaced by \( \epsilon(p)p^{k-1} \). In this paper, we construct such a distribution for the case when \( f \) is a CM modular form that is non-ordinary at \( p \). In other words, when the \( L \)-function of \( f \) coincides with that of a Grossencharacter \( \phi \) defined over \( K \) and \( p \) inerts in \( K \). More precisely, we prove the following theorem in §3 (Theorem 3.20).

Theorem 1.1. If \( f \) is as above, then there exist even admissible distributions \( \mu_{\pm \epsilon(p)p^{k-1}} \) such that

\[
\int_{\mathbb{Z}_p^*} \theta d\mu_{\pm \epsilon(p)p^{k-1}} = \frac{p^{3n(k-1)}}{(\pm \epsilon(p)p^{k-1})^{\tau(\theta^{-1})}} \times \frac{L(\text{Sym}^2 f,\theta^{-1},2k-2)}{(\text{period})}.
\]

Note that we have \( \alpha_1(p)^2 = \alpha_2(p)^2 = -\epsilon(p)p^{k-1} \) in this case, methods in [3] only produce one distribution, which agrees with \( \mu_{-\epsilon(p)p^{k-1}} \) as given by Theorem 1.1.

The idea of the construction is rather simple. Let \( V_f \) be the \( p \)-adic representation of \( G_Q \) associated to \( f \) as constructed by Deligne in [4]. In order to prove Theorem 1.1, we make use of the following observation. As \( G_Q \)-representations, we have

\[ \text{Sym}^2(V_f) \cong V_1 \oplus V_2 \]

where \( V_1 \) is an one-dimensional representation associated to some Dirichlet character \( \eta \) twisted by a power of the cyclotomic character and \( V_2 \) is a two-dimensional representation associated to the Grossencharacter \( \phi^2 \). This implies that the \( L \)-function of \( f \) factorises into

\[ L(\text{Sym}^2 f,s) = L(\phi^2,s) L(\eta,s-k+1). \]
We can therefore make use of an Euler system constructed from elliptic units to interpolate the \( L \)-values of \( \phi^2 \) and multiply the resulting distributions with an appropriate twist of the Kubota-Leopoldt \( p \)-adic \( L \)-function associated to \( \eta \), which interpolates the \( L \)-values of \( \eta \).

Because of the symmetry between the two distributions, we show that some plus and minus logarithms \( \log^+ \) of Pollack divide \( \mu_+ (p^{k-1}) \pm \mu_- (p^{k-1}) \). This allows us to obtain two bounded measures:

**Theorem 1.2.** (Theorem 3.25) Let \( \theta \) be an even Dirichlet character of conductor \( p^n \). There exist bounded \( p \)-adic measures \( \mu^\pm (\text{Sym}^2 (V_f)) \) such that the followings hold.

(a) If \( n \) is even, then

\[
\int_{\mathbb{Z}_p^\times} \theta \mu^+ (\text{Sym}^2 (V_f)) = \frac{(2k - 3)! (k - 1)! p^{n(k - 1)}}{\theta (\log^+ ) \tau (\theta^{-1})^2 \varepsilon (p)^n} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k - 2)}{\text{period}};
\]

(b) If \( n \) is odd, then

\[
\int_{\mathbb{Z}_p^\times} \theta \mu^- (\text{Sym}^2 (V_f)) = \frac{(2k - 3)! (k - 1)! p^{n(k - 1)}}{\theta (\log^- ) \tau (\theta^{-1})^2 \varepsilon (p)^n} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k - 2)}{\text{period}}.
\]

Moreover, \( \mu^\pm (\text{Sym}^2 (V_f)) \) are uniquely determined by (a) and (b) respectively.

In §4, we make use of some of the ideas in [6] to show that these measures can be obtained from some appropriate Coleman maps and define the corresponding plus and minus Selmer groups \( \text{Sel}_p^\pm (\text{Sym}^2 (V_f)) \). On identifying the measures as elements in some Iwasawa algebra \( \Lambda \otimes \mathbb{Q} \), we show that the following holds under some appropriate conditions (see Theorem 4.8 for a precise statement):

**Theorem 1.3.** The Selmer groups \( \text{Sel}_p^\pm (\text{Sym}^2 (V_f)) \) are \( \Lambda \)-cotorsion and

\[
\text{Char}_{\Lambda \otimes \mathbb{Q}} (\text{Sel}_p^\pm (\text{Sym}^2 (V_f))) = \left( \mu^\pm (\text{Sym}^2 (V_f)) \right).
\]

Finally, in the appendix, we explain how some of the linear algebra results that we use to prove the main theorems can be easily generalised to general symmetric powers \( \text{Sym}^m f \) where \( m \geq 2 \) is an integer.

2. Notation

2.1. Extensions by \( p \) power roots of unity. Throughout this paper, \( p \) is an odd prime. If \( K \) is a field of characteristic 0, either local or global, \( G_K \) denotes its absolute Galois group, \( \chi \) the \( p \)-cyclotomic character on \( G_K \) and \( O_K \) the ring of integers of \( K \). We write \( \iota \) for the complex conjugation in \( G_Q \).

For an integer \( n \geq 0 \), we write \( K_n \) for the extension \( K(\mu_{p^n}) \) where \( \mu_{p^n} \) is the set of \( p^n \)-th roots of unity and \( K_{\infty} \) denotes \( \bigcup_{n \geq 1} K_n \). When \( K = \mathbb{Q} \), we write \( k_n = \mathbb{Q}(\mu_{p^n}) \) instead. In particular, we write \( Q_{p,n} = Q_p(\mu_{p^n}) \).

Let \( G_n \) denote the Galois group \( \text{Gal}(Q_{p,n}/Q_p) \) for \( 0 \leq n \leq \infty \). Then, \( G_{\infty} \cong \Delta \times \Gamma \) where \( \Delta = G_1 \) is a finite group of order \( p - 1 \) and \( \Gamma = \text{Gal}(Q_{p,\infty}/Q_{p,1}) \cong \mathbb{Z}/p \). We fix a topological generator \( \gamma \) of \( \Gamma \).

2.2. Iwasawa algebras and power series. Given a finite extension \( K \) of \( Q_p, L_{K}(G_{\infty}) \) (respectively \( \Delta L_{K}(\Gamma) \)) denotes the Iwasawa algebra of \( G_{\infty} \) (respectively \( \Gamma \)) over \( O_K \). We write \( L_{K}(G_{\infty}) = L_{O_K}(G_{\infty}) \otimes K \) and \( L_{K}(\Gamma) = L_{O_K}(\Gamma) \otimes K \). If \( M \) is a finitely generated \( L_{O_K}((\Gamma)) \)-torsion (respectively \( L_{K}(\Gamma) \)-torsion) module, we write \( \text{Char}_{L_{O_K}((\Gamma))}(M) \) (respectively \( \text{Char}_{L_{K}(\Gamma)}(M) \)) for its characteristic ideal.

Given a module \( M \) over \( L_{O_K}(G_{\infty}) \) (respectively \( L_{K}(G_{\infty}) \)) and a character \( \delta : \Delta \to \mathbb{Z}/p^\times \), \( M^\delta \) denotes the \( \delta \)-isotypical component of \( M \). For any \( m \in M \), we write \( m^\delta \) for the projection of \( m \) into \( M^\delta \). The Pontryagin dual of \( M \) is written as \( M^\vee \).

Let \( r \in \mathbb{R}_{\geq 0} \). We define

\[
\mathcal{H}_r = \bigg\{ \sum_{n \geq 0, \sigma \in \Delta} c_{n, \sigma} \cdot \sigma \cdot X^n \in \mathbb{C}_p[[X]] : \sup_n \frac{|c_{n, \sigma}|}{n^r} < \infty \ \forall \sigma \in \Delta \bigg\}
\]

where \( | \cdot |_p \) is the \( p \)-adic norm on \( \mathbb{C}_p \) such that \( |p|_p = p^{-1} \). We write \( \mathcal{H}_{\infty} = \bigcup_{r \geq 0} \mathcal{H}_r \) and \( \mathcal{H}_r(G_{\infty}) = \{ f(\gamma - 1) : f \in \mathcal{H}_r \} \) for \( r \in \mathbb{R}_{\geq 0} \cup \{ \infty \} \). In other words, the elements of \( \mathcal{H}_r \) (respectively \( \mathcal{H}_r(G_{\infty}) \)) are the power series...
in $X$ (respectively $\gamma - 1$) over $\mathbb{C}_p[\Delta]$ with growth rate $O(\log p^n)$. If $F,G \in \mathcal{H}_\infty$ or $\mathcal{H}_\infty(G_\infty)$ are such that $F = O(G)$ and $G = O(F)$, we write $F \sim G$.

Given a subfield $K$ of $\mathbb{C}_p$, we write $\mathcal{H}_{K,R} = \mathcal{H}_r \cap K[\Delta][[X]]$ and similarly for $\mathcal{H}_{K,R}(G_\infty)$. In particular, $\mathcal{H}_{K,R}(G_\infty) = \Lambda_K(G_\infty)$.

Let $n \in \mathbb{Z}$. We define the $K$-linear map $\text{T}_{\nu,n}$ from $\mathcal{H}_{K,R}(G_\infty)$ to itself to be the map that sends $\sigma$ to $\chi(\sigma)^n \sigma$ for all $\sigma \in G_\infty$. It is clearly bijective (with inverse $\text{T}_{\nu,-n}$).

2.3. Crystalline representations. We write $\mathcal{B}_{\text{cris}}$ and $\mathcal{B}_{\text{dR}}$ for the rings of Fontaine and $\varphi$ for the Frobenius acting on these rings. Recall that there exists an element $t \in \mathcal{B}_{\text{dR}}$ such that $\varphi(t) = pt$ and $g \cdot t = \chi(g)t$ for $g \in G_{\overline{\mathbb{Q}}}$.

Let $V$ be a $p$-adic representation of $G_{\overline{\mathbb{Q}}}$, we denote the Dieudonné module by $\mathcal{D}_{\text{cris}}(V) = (\mathcal{B}_{\text{cris}} \otimes V)^{G_{\overline{\mathbb{Q}}}}$. We say that $V$ is crystalline if $V$ has the same $\mathbb{Q}_p$-dimension as $\mathcal{D}_{\text{cris}}(V)$. Fix such a $V$. If $j \in \mathbb{Z}$, $\mathcal{D}_{\text{cris}}^j(V)$ denotes the $j$-th de Rham filtration of $\mathcal{D}_{\text{cris}}(V)$.

Let $T$ be a lattice of $V$ which is stable under $G_{\overline{\mathbb{Q}}}$, let $\mathcal{H}^j_{\text{tate}}(T)$ denote the inverse limit $\lim_{\longrightarrow} H^j_\text{dR}(\mathcal{Z}[\mu_p^n],1/p,T)$ with respect to the corestriction and $H^1_\text{tate}(V) = \mathbb{Q} \otimes \mathcal{H}^1_{\text{tate}}(T)$. Moreover, if $V$ arises from a restriction of a $p$-adic representation of $G_{\mathbb{Q}}$ and $T$ is a lattice stable under $G_{\mathbb{Q}}$, we write

$$\mathcal{H}^j(T) = \lim_{\longrightarrow} H^j_\text{dR}(\mathcal{Z}[\mu_p^n],1/p,T) \quad \text{and} \quad \mathcal{H}^j(V) = \mathbb{Q} \otimes \mathcal{H}^1(T).$$

We have localisation maps

$$\text{loc} : \mathcal{H}^1(T) \to \mathcal{H}^1_{\text{tate}}(T) \quad \text{and} \quad \text{loc} : \mathcal{H}^1(V) \to \mathcal{H}^1_{\text{tate}}(V).$$

If $F$ is a number field, we define the $p$-Selmer group of $T$ over $F$ to by

$$\text{Sel}_p(T/F) = \ker \left( H^1(K,T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \to \prod_v H^1(F_v,T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \right)$$

where $v$ runs through the places of $F$.

Let $V(j)$ denote the $j$-th Tate twist of $V$, i.e. $V(j) = V \otimes \mathbb{Q}_p e_j$ where $G_{\overline{\mathbb{Q}}}$ acts on $e_j$ via $\chi^j$. We have

$$\mathcal{D}_{\text{cris}}(V(j)) = t^{-j} \mathcal{D}_{\text{cris}}(V) \otimes e_j.$$

For any $v \in \mathcal{D}_{\text{cris}}(V)$, $v_j = v \otimes t^{-j} e_j$ denotes its image in $\mathcal{D}_{\text{cris}}(V(j))$. We write $\text{T}_{\nu,j} : \mathcal{H}^1_{\text{tate}}(V) \to \mathcal{H}^1_{\text{tate}}(V(j))$ for the isomorphism defined in [9, § A.4], which depends on a choice of primitive $p$-power roots of unity.

Finally, we write

$$\text{exp} : \mathbb{Q}_p \otimes \mathcal{D}_{\text{cris}}(V) \to H^1(Qp,\mathbb{V},V) \quad \text{and} \quad \text{exp}^* : H^1(Qp,\mathbb{V},V) \to \mathbb{Q}_p \otimes \mathcal{D}_{\text{cris}}(V)$$

for Bloch-Kato’s exponential and dual exponential respectively.

2.4. Imaginary quadratic fields. Let $K$ be an imaginary quadratic field with ring of integers $O$ and idele class group $C_K$. We write $\varepsilon_K$ for the quadratic character associated to $K$, i.e. the character on $G_{\mathbb{Q}}$ which sends $\sigma$ to 1 if $\sigma \in G_K$ and to $-1$ otherwise.

A Grossencharacter of $K$ is simply a continuous homomorphism $\phi : C_K \to C^\times$ with complex $L$-function

$$L(\phi,s) = \prod_v (1 - \phi(v)N(v)^{-s})^{-1}$$

where the product runs through the finite places $v$ of $K$ at which $\phi$ is unramified, $\phi(v)$ is the image of the uniformiser of $K_v$ under $\phi$ and $N(v)$ is the norm of $v$. Let $f$ be the conductor of $\phi$. We say that $\gamma$ is of type $(m,n)$ if $m,n \in \mathbb{Z}$ if the restriction of $\eta$ to the archimedean part $C^\times$ of $C_K$ is of the form $z \mapsto z^m z^n$.

We write $K = \mathcal{O}_K(p^m f)$, $K(\mathcal{a})$ denotes the ray class field of $K$ modulo $\mathcal{a}$ if $\mathcal{a}$ is an ideal of $O$.

If $T$ is a $\mathbb{Z}_p$-representation of $G_K$, we write

$$\mathcal{H}^1_{p \times f}(T) = \lim_{K'} H^1(Q_K[I/p],T) \quad \text{and} \quad \mathcal{H}^1_{p \times f}(Q \otimes \mathbb{Z}_p) = \mathcal{H}^1_{p \times f}(T) \otimes \mathbb{Z}_p \mathbb{Q}$$

where $K'$ ranges over all finite extensions of $K$ contained in $K(p^\infty f)$. 

2.5. Modular forms. Let $f = \sum a_n q^n$ be a normalised eigen-newform of weight $k \geq 2$, level $N$ and character $\epsilon$. We assume that $f$ is a CM modular form, i.e. $L(f, s) = L(\phi, s)$ for some Grossencharacter $\phi$ of an imaginary quadratic field $K$ with conductor $\mathfrak{f}$. Then, $\phi$ is of type $(-k + 1, 0)$. Moreover, $p$ inerts in $K$ if and only if $f$ is non-ordinary at $p$. In this case, $a_p$ is always 0. Throughout, we fix such a $p$ with $p \neq 2$.

The coefficient field $F_f$ of $f$ is contained in the field of definition of $\phi$. We write $E$ for the completion of this field at a fixed prime above $p$.

We write $V_f$ for the 2-dimensional $E$-linear representation of $G_\mathbb{Q}$ associated to $f$ from [4], so we have a homomorphism

$$\rho_f : G_\mathbb{Q} \to \text{GL}(V_f).$$

Throughout the paper, we assume that the following hypothesis holds.

**Hypothesis 2.1.** If $\epsilon$ and $K$ are as above, then $\epsilon_K \neq \epsilon$.

3. $p$-adic $L$-functions

3.1. Grossencharacters over $K$. We first review some results on Grossencharacters. Let $\eta$ be a Grossencharacter on $G_K$ of conductor $\mathfrak{f}$. We fix a finite extension $E$ of $\mathbb{Q}_p$ such that $E$ contains the image of $\eta$. We write $V(\eta)$ for the one-dimensional $E$-linear representation of $G_K$. It is a representation that factors through $\text{Gal}(K/K)$. For an ideal $\mathfrak{a}$ of $\mathcal{O}$ which is prime to $\mathfrak{p}f$, the Artin symbol $\left(\mathfrak{a}, K/K\right) \in \text{Gal}(K/K)$ acts on $V(\eta)$ as the multiplication by $\eta(\mathfrak{a})^{-1}$. We write $\tilde{\eta} : G_K \to E^\times$ for the corresponding character.

We write $\tilde{V}_\eta = \text{Ind}_K^\mathbb{Q}(V(\eta))$. The canonical homomorphism $K \otimes \mathbb{Q}_p(\epsilon_K) \to K(p^{\infty}\mathfrak{f})$ induces a map

$$\text{Ind} : H^1_{p^{\infty}\mathfrak{f}}(V(\eta)) \to H^1(\tilde{V}_\eta).$$

Let $\gamma$ be a non-zero element of $V(\eta)$. By [5, §15.5], a system of norm compatible elliptic units in $K(p^{\infty}\mathfrak{f})$ defines an element $z_{p^{\infty}\mathfrak{f}} \in H^1_{p^{\infty}\mathfrak{f}}(\mathbb{Z}_p(1))$. We write the image of $z_{p^{\infty}\mathfrak{f}}$ under the composition

$$H^1_{p^{\infty}\mathfrak{f}}(\mathbb{Z}_p(1)) \xrightarrow{\gamma} H^1_{p^{\infty}\mathfrak{f}}(V(\eta)(1)) \xrightarrow{\text{Ind}} H^1(\tilde{V}_\eta(1)) \xrightarrow{\text{loc}} H^1_{\text{loc}}(\tilde{V}_\eta(1)) \xrightarrow{\text{Tw}} H^1(\tilde{V}_\eta)$$

as $z_\gamma(\eta) = z(\eta)$ and its projection into $H^1(\mathbb{Q}_p, \tilde{V}_\eta(j))$ is denoted by $z_{1, \eta}(\eta)$.

Note that the eigenvalues of $\iota$ on $\tilde{V}_\eta$ are $\pm 1$, each with multiplicity 1. If $v \in \tilde{V}_\eta$, we write $v^\pm$ for the projection of $v$ into the $\pm 1$-eigenspace.

**Proposition 3.1.** Let $\eta$ be a Grossencharacter over $K$ of type $(-r, 0)$ with $r \geq 1$. Let $\theta$ be a character on $G_n$ and write

$$\kappa_\theta : \mathbb{Q}_{p, n} \otimes \text{Fil}^0 \mathcal{D}_{\text{cris}}(\tilde{V}_\eta(1)) \to \mathbb{C} \otimes \tilde{V}_\eta(1)$$

$$x \otimes y \mapsto \sum_{\sigma \in G_n} \theta(\sigma)\sigma(x) \text{per}(y)$$

where per is the period map associated to $\eta$ as defined in [5, §15.8]. Then, we have

$$\kappa_\theta \circ \text{exp}^*(z_{1, \eta}(\eta)) = L_{(p)}(\eta^\theta, r) \cdot (\gamma')^\pm$$

where $\pm = \theta(-1)$ and $\gamma'$ denotes the image of $\gamma$ in $\tilde{V}_\eta$.

**Proof.** [5, §15.12].

3.2. The symmetric square of a CM modular form. Let $f$ be a modular form as in §2.5. By comparing the eigenvalues of Frobenii, we see that the representation $V_f$ is isomorphic to $V_\phi = \text{Ind}_K^\mathbb{Q} V(\phi)$. Therefore, $V_f$ admits a basis $x, y$ such that for $\sigma \in G_\mathbb{Q}$, the matrix of $\rho_f(\sigma)$ with respect to this basis is given by

$$\rho_f(\sigma) = \begin{pmatrix} \phi(\sigma) & 0 \\ 0 & \phi(\sigma^t) \end{pmatrix}$$

if $\sigma \in G_K$. Otherwise,

$$\rho_f(\sigma) = \begin{pmatrix} 0 & \phi(\sigma^t) \\ \phi(\sigma) & 0 \end{pmatrix}$$
where \( \sigma = \iota \sigma' \) with \( \sigma' \in G_K \).

**Lemma 3.2.** The determinant of \( \rho_f \) is given by
\[
\det(\rho_f)(\sigma) = \begin{cases} 
\phi(\sigma)\phi(\iota \sigma t) & \text{if } \sigma \in G_K \\
-\phi(\sigma')\phi(\iota \sigma' t) & \text{if } \sigma = \iota \sigma' \text{ where } \sigma' \in G_K.
\end{cases}
\]

**Proof.** This is immediate from (2) and (3). \( \square \)

**Proposition 3.3.** As a \( G_Q \)-representation, \( \text{Sym}^2(V_f) \) decomposes into
\[
\text{Sym}^2(V_f) \cong V_1 \oplus V_2
\]
where \( \rho_i : G_Q \to \text{GL}(V_i) \) is an \( i \)-dimensional representation of \( G_Q \) for \( i = 1, 2 \). Moreover,
\[
\begin{align*}
\rho_1 & \cong \varepsilon_K \cdot \det(\rho_f) = \varepsilon_K \cdot \varepsilon \cdot \chi^{k-1}, \\
\rho_2 & \cong \tilde{V}_{\phi^2}.
\end{align*}
\]

**Proof.** It is clear that \( x \otimes x, y \otimes y, x \otimes y + y \otimes x \) form a basis of \( \text{Sym}^2(V_f) \). By formulae (2) and (3), \( \sigma \cdot (x \otimes y + y \otimes x) \) is a multiple of \( x \otimes y + y \otimes x \) for any \( \sigma \in G_Q \). Hence, it gives an one-dimensional sub-representation \( V_1 \) of \( \text{Sym}^2(V_f) \). More explicitly, we have
\[
\sigma \cdot (x \otimes y + y \otimes x) = \begin{cases} 
\phi(\sigma)\phi(\iota \sigma t)(x \otimes y + y \otimes x) & \text{if } \sigma \in G_K \\
-\phi(\sigma')\phi(\iota \sigma' t)(x \otimes y + y \otimes x) & \text{if } \sigma = \iota \sigma' \text{ where } \sigma' \in G_K.
\end{cases}
\]

Therefore, we deduce (4) from Lemma 3.2.

It is also clear that \( x \otimes x, y \otimes y \) form a basis of a 2-dimensional representation \( \rho_2 : G_Q \to \text{GL}(V_2) \). With respect to this basis,
\[
\rho_2(\sigma) = \begin{pmatrix} \phi^2(\sigma) & 0 \\
0 & \phi^2(\iota \sigma t) \end{pmatrix}
\]
if \( \sigma \in G_K \). Otherwise, if \( \sigma = \iota \sigma' \) where \( \sigma' \in G_K \), then
\[
\rho_2(\sigma) = \begin{pmatrix} 0 & \phi^2(\iota \sigma' t) \\
\phi^2(\sigma') & 0 \end{pmatrix}.
\]

Therefore, \( V_2 \cong \text{Ind}_K^G V(\phi^2) \) as required. \( \square \)

**Corollary 3.4.** The complex \( L \) function admits a factorisation
\[
L(\text{Sym}^2 f, s) = L(\phi^2, s)L(\varepsilon_K \cdot \varepsilon, s-k+1).
\]

**Proof.** The \( L \)-function of \( \text{Sym}^2 f \) only have non-trivial Euler factors at \( q \nmid N \). The Euler factors on the two sides of the equation at \( q \) agree by Proposition 3.3, so we are done. \( \square \)

### 3.3. The symmetric square as a \( G_{Q_p} \)-representation

We study the representation \( \text{Sym}^2(V_f) \) restricted to \( G_{Q_p} \). More specifically, we study \( \mathbb{D}_{\text{cris}}(\text{Sym}^2 V_f) \).

**Lemma 3.5.** As \( G_{Q_p} \)-representations, both \( V_1 \) and \( V_2 \) are crystalline.

**Proof.** The functor \( \mathbb{D}_{\text{cris}} \) is compatible with taking direct sums, so we can identify \( \mathbb{D}_{\text{cris}}(V_i) \) as a filtered sub-\( \varphi \)-module of \( \mathbb{D}_{\text{cris}}(V_f) \) for \( i = 1, 2 \). That is,
\[
\mathbb{D}_{\text{cris}}(\text{Sym}^2(V_f)) \cong \mathbb{D}_{\text{cris}}(V_1) \oplus \mathbb{D}_{\text{cris}}(V_2).
\]

Since \( \text{Sym}^2(V_f) \) is crystalline, so \( \mathbb{D}_{\text{cris}}(\text{Sym}^2(V_f)) \) is of dimension 3 over \( E \). Hence, \( \mathbb{D}_{\text{cris}}(V_i) \) must have dimension \( i \) and \( V_i \) is crystalline for \( i = 1, 2 \). \( \square \)
We now give explicit descriptions of $\mathcal{D}_{\text{cris}}(V_1)$ and $\mathcal{D}_{\text{cris}}(V_2)$.

Recall that $\mathcal{D}_{\text{cris}}(V_f)$ is a 2-dimensional $E$-vector space with Hodge-Tate weights 0 and $1 - k$. Moreover, the de Rham filtration is given by

$$\text{Fil}^i\mathcal{D}_{\text{cris}}(V_f) = \begin{cases} E\omega \oplus E\varphi(\omega) & \text{if } i \leq 0 \\ E\omega & \text{if } 1 \leq i \leq k - 1 \\ 0 & \text{if } i \geq k \end{cases}$$

for some $\omega \neq 0$. The action of $\varphi$ on $\mathcal{D}_{\text{cris}}(V_f)$ satisfies $\varphi^2 = -\varepsilon(p)pk^{-1}$. Therefore,

$$\text{Fil}^i\mathcal{D}_{\text{cris}}(\text{Sym}^2(V_f)) = \begin{cases} \mathcal{D}_{\text{cris}}(\text{Sym}^2(V_f)) & \text{if } i \leq 0 \\ E(\omega \otimes \omega) \oplus E(\varphi(\omega) \otimes \omega) & \text{if } 1 \leq i \leq k - 1 \\ E(\omega \otimes \omega) & \text{if } k \leq i \leq 2k - 2 \\ 0 & \text{if } i \geq 2k - 1 \end{cases}$$

Since $\varphi^2(\omega) = -\varepsilon(p)pk^{-1}\omega$, we have

$$\varphi\left(\omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega\right) = -\varepsilon(p)pk^{-1}\left(\omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega\right).$$

In particular, $\omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega$ is an eigenvector of $\varphi$. Therefore, we have a decomposition of filtered $\varphi$-modules

$$\mathcal{D}_{\text{cris}}(\text{Sym}^2(V_f)) = \left(E(\omega \otimes \omega) \oplus E(\varphi(\omega) \otimes \varphi(\omega))\right) \oplus \left(E(\omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega)\right).$$

**Proposition 3.6.** As filtered $\varphi$-modules, we have

$$\mathcal{D}_{\text{cris}}(V_1) = E(\varphi(\omega) \otimes \omega + \omega \otimes \varphi(\omega)), \quad \mathcal{D}_{\text{cris}}(V_2) = E(\omega \otimes \omega) \oplus E(\varphi(\omega) \otimes \varphi(\omega)).$$

**Proof.** By (4), $\rho_1 = \varepsilon_K \cdot \varphi^{k-1}$. Since $p$ is inert in $K$, $\varepsilon_K(p) = -1$. The Hodge-Tate weight of $V_1$ is therefore $1 - k$ and $\varphi$ acts on $\mathcal{D}_{\text{cris}}(V_1)$ as multiplication by $-\varepsilon(p)pk^{-1}$. This proves the first equality. The second equality is then automatic by (6). $\square$

**Remark 3.7.** Such a decomposition of $G_{\mathbb{Q}_p}$-representations is in fact possible for $f$ without CM (see [10, §2.2]).

**Corollary 3.8.** The eigenvalues of $\varphi$ on $\mathcal{D}_{\text{cris}}(V_2)$ are $\pm \varepsilon(p)pk^{-1}$.

**Proof.** By Proposition 3.6, the matrix of $\varphi$ with respect to the basis $\omega \otimes \omega, \varphi(\omega) \otimes \varphi(\omega)$ is

$$\begin{pmatrix} 0 & \varepsilon(p)pk^{-2} \\ 1 & 0 \end{pmatrix},$$

hence the result. $\square$

**Corollary 3.9.** The Hodge-Tate weights of $V_2$ are 0 and $2 - 2k$.

**Proof.** This follows from (8) and Proposition 3.6. $\square$

### 3.4 The Perrin-Riou pairing

By Corollary 3.8, the slope of $\varphi$ on $\mathcal{D}_{\text{cris}}(V_2)$ is $k - 1$. Hence, by Corollary 3.9, given any $v \in \mathcal{D}_{\text{cris}}(V_2)$, we have the Perrin-Riou pairing

$$\mathcal{L}_v : \mathcal{H}^1_{\text{cor}}(V_2) \to \mathcal{H}_{k-1,E}(G_{\infty})$$

which satisfies the following properties.

**Proposition 3.10.** For an integer $r \geq 0$, we have

$$\chi^r(\mathcal{L}_v(z)) = r! \left[ 1 - \frac{\varphi^{-1}}{p} \right] (1 - \varphi)^{-1}(v_{r+1}), \exp^*(z_{r,0}) \right]_0.$$
Let \( \theta \) be a character of \( G_n \) which does not factor through \( G_{n-1} \) with \( n \geq 1 \), then
\[
\chi^\prime \theta \left( \mathcal{L}_e(z) \right) = \frac{r!}{\tau(\theta^{-1})} \sum_{\sigma \in G_n} \theta^{-1}(\sigma) \left[ \varphi^{-n}(z^{n+1}), \exp(z^{n-1}) \right]_n
\]
where \([,]_n\) is the pairing
\[
[,]_n : H^1(\mathbb{Q}_p, n, V_2(r + 1)) \times H^1(\mathbb{Q}_p, n, V_2^*(-r)) \to H^2(\mathbb{Q}_p, n, E(1)) \equiv E,
\]
\(z_{-r,n}\) denotes the projection of \( \operatorname{Tw}_{-r}(z) \) into \( H^1(\mathbb{Q}_p, n, V_2^*(-r)) \) and \( \tau(\theta^{-1}) \) denotes the Gauss sum of \( \theta^{-1} \).

**Proof.** See [6, §3.2]. \( \square \)

**Remark 3.11.** The assumption on the eigenvalues of \( \varphi \) made in [6] are not necessary for our purposes here because the Perrin-Riou pairings can be defined by applying \( 1 - \varphi \) to the \((\varphi, G_{\infty})\)-module of \( V_2^* \) (see [7] and [5, §16.4]).

We fix a non-zero element \( \bar{\omega} \in \operatorname{Fil}^{-1} \mathbb{D}_{\text{cris}}(V_2^*(1)) \) and write
\[
\operatorname{per}(\bar{\omega}) = \Omega_+ (\gamma')^+ + \Omega_-(\gamma')^-,
\]
where \( \Omega_\pm \in \mathbb{C}^x \) and \( \gamma' \) is as in the statement of Proposition 3.1 for some fixed \( \gamma \).

**Definition 3.12.** Under the choices made above, we define \( v^\pm \in \mathbb{D}_{\text{cris}}(V_2) \) by
\[
v^\pm = \frac{1}{[\varphi(\omega) \otimes \varphi(\omega), \bar{\omega}]} \left( \pm \epsilon(p)p^{k-1} \omega \otimes \varphi(\omega) \otimes \varphi(\omega) \right).
\]

**Lemma 3.13.** The elements \( v^\pm \) satisfy:

(a) Both \( v^\pm \) are eigenvalues of \( \varphi \) with \( \varphi(v^\pm) = \pm \epsilon(p)p^{k-1}v^\pm \);

(b) For any \( x \in \operatorname{Fil}^0 \mathbb{D}_{\text{cris}}(V_2^*(-r)) \) and an integer \( r \) such that \( 0 \leq r \leq 2k - 3 \), we have
\[
[v^+_{r+1}, x] = [v^-_{r+1}, x]
\]
where \([,] : \mathbb{D}_{\text{cris}}(V_2(r + 1)) \times \mathbb{D}_{\text{cris}}(V_2^*(-r)) \to \mathbb{D}_{\text{cris}}(E(1)) = E \cdot t^{-1}e_1 \).

**Proof.** (a) is easy to check using the matrix given in the proof of Corollary 3.8 (or by direct calculations).

By Corollary 3.9, the Hodge-Tate weights of \( V_2^* \) are 0 and 2k - 2. Hence, \( \operatorname{Fil}^0 \mathbb{D}_{\text{cris}}(V_2^*(-r)) \) is one-dimensional with basis \( \bar{\omega}_{-r-1} \) for \( 0 \leq r \leq 2k - 3 \). Since \( (\omega \otimes \omega)_{r+1} \in \operatorname{Fil}^0 \mathbb{D}_{\text{cris}}(V_2(r + 1)) \), we have
\[
[(\omega \otimes \omega)_{r+1}, \bar{\omega}_{-r-1}] = 0.
\]
Hence,
\[
[v^+_{r+1}, \bar{\omega}_{-r-1}] = [v^-_{r+1}, \bar{\omega}_{-r-1}] = 1,
\]
which implies (b). \( \square \)

Note that \( V_2^* \cong V_{\phi^2}(2k - 2) \). This enables us to make the following definition of \( p \)-adic \( L \)-functions associated to \( \phi^2 \).

**Definition 3.14.** On taking \( \eta = \bar{\phi}^2 \) in §3.1, we define
\[
L_{\chi \cdot (p)p^{k-1}}(\phi^2) = \mathcal{L}_{\phi^2} \left( \operatorname{Tw}_{2k-2} \left( z(\phi^2) \right) \right) \in \mathcal{H}_{k-1,E}(G_{\infty}).
\]

**Lemma 3.15.** Let \( \theta \) be a character of \( G_n \) which does not factor through \( G_{n-1} \) with \( n \geq 1 \) and write \( \delta = \theta^{-1} \), then
\[
\chi^{2k-3} \theta \left( L_{\alpha}(\phi^2) \right) = \frac{(2k - 3)!p^{(2k-2)n}}{\tau(\theta^{-1})a^n} \times \frac{L(\phi^2\theta^{-1}, 2k - 2)}{\Omega_\delta}
\]
where \( \alpha = \pm \epsilon(p)p^{k-1} \).
Proof. We have
\[
\chi^{2k-3}\theta\left(L_{\pm1(p)p^{k-1}}(\phi^2)\right) = \chi^{2k-3}\theta\left(L_{\pm1}(Tw_{2k-2}(z(\phi^2)))\right) = \frac{(2k-3)!}{\tau(\theta^{-1})}\left[\sum_{\sigma \in G_n}\theta^{-1}(\sigma)\exp^*(z_{1,n}(\phi^2)^\sigma)\right]_n
\]
where the second equality follows from Proposition 3.10, the third follows from Lemma 3.13(a) and the last equality is a consequence of Proposition 3.1 and the fact that \(p\) divides the conductor of \(\theta\). \(\Box\)

Lemma 3.16. We have
\[
\chi^{2k-3}\left(L_{\pm1(p)p^{k-1}}(\phi^2)\right) = \left(1 - p^{-1} + (1 - \epsilon(p)^{-2}p^{2k-3})(\pm \epsilon(p)p^{1-k})\right) \frac{L(\phi^2, 2k - 2)}{\Omega^+}.
\]

Proof. Since \(\varphi^2 = \epsilon(p)^2p^{2k-2}\) on \(\mathbb{D}_{\text{cris}}(V_2(2k - 2))\), we have
\[
\begin{align*}
1 - \frac{\varphi^{-1}}{p} & = \left(1 - \epsilon(p)^{-2}p^{2k-3}\varphi\right) \frac{1 + \varphi}{1 - \epsilon(p)^2p^{2k-2}} = \frac{1 - p^{-1} + (1 - \epsilon(p)^{-2}p^{2k-3})\varphi}{1 - \epsilon(p)^2p^{2k-2} + \epsilon(p)^2p^{2k-2} - 2k}.
\end{align*}
\]
Therefore, similarly to the proof of Lemma 3.15, we have
\[
\chi^{2k-3}\left(L_{\pm1(p)p^{k-1}}(\phi^2)\right) = \chi^{2k-3}\left(L_{\pm1}(Tw_{2k-2}(z(\phi^2)))\right) = (2k-3)! \left[1 - p^{-1} + (1 - \epsilon(p)^{-2}p^{2k-3})\varphi\right]_0 \frac{L_\delta(\phi^2, 2k - 2)}{\Omega^+} \left(1 - \epsilon(p)^2p^{2k-2} + \epsilon(p)^2p^{2k-2} - 2k\right)
\]
\[
= \left(1 - p^{-1} + (1 - \epsilon(p)^{-2}p^{2k-3})(\pm \epsilon(p)p^{1-k})\right) \frac{L(\phi^2, 2k - 2)}{\Omega^+} \left(1 - \epsilon(p)^2p^{2k-2} + \epsilon(p)^2p^{2k-2} - 2k\right)
\]
\(\Box\)

Remark 3.17. Consider the \(p\)-adic L-function \(L_{\pm1(p)p^{k-1}}(\phi^2)\). The first factor on the right-hand side of the equation in the statement of Lemma 3.16 vanishes if and only if \(k = 2\) and \(\epsilon(p) = 1\) (e.g. when \(f\) corresponds to an elliptic curve over \(\mathbb{Q}\)). This recovers the trivial zero result in [10].
3.5. p-adic L-functions of the symmetric square. Let us first recall the following result of Kubota and Leopoldt.

**Theorem 3.18.** If \( \eta \) is a non-trivial Dirichlet character of conductor prime to \( p \), there exists a bounded p-adic measure \( L_p(\eta) \in \mathcal{H}_{0,E}(G_\infty) \) where \( E \) is some finite extension of \( \mathbb{Q}_p \) which contains the image of \( \eta \) such that

\[
\chi^r \theta(L_p(\eta)) = \frac{(r+1)!p^n(r+1)}{(2\pi i)^{r+1}} \times L(\eta \theta^{-1}, r+1);
\]

\[
\chi^r(L_p(\eta)) = \frac{(r+1)!}{(2\pi i)^n+1} L(\eta, r+1).
\]

for any integer \( r \geq 0 \) and Dirichlet character \( \theta \) of conductor \( p^n \) such that \( \chi^{r+1}\theta(-1) = \eta(-1) \).

Since we assume that Hypothesis 2.1 holds, we may take \( \eta = \varepsilon_K \cdot \epsilon \) in Theorem 3.18. This enables us to give the following definition.

**Definition 3.19.** For \( \alpha = \pm \epsilon(p)p^{k-1} \) we define

\[
L_\alpha(\text{Sym}^2(V_f)) = L_\alpha(\phi^2) \times \text{Tw}_{-k+1}(L_p(\varepsilon_K \cdot \epsilon)).
\]

For the rest of this section, unless otherwise stated, \( \theta \) denotes an even character on \( G_n \) which does not factor through \( G_{n-1} \) with \( n \geq 1 \).

**Theorem 3.20.** Both \( L_{\lambda,\epsilon(p)p^{k-1}}(\text{Sym}^2(V_f)) \) lie inside \( \mathcal{H}_{k-1,E}(G_\infty) \) and admit the following interpolating properties:

\[
\chi^{2k-3} \left( L_\alpha(\text{Sym}^2(V_f)) \right) = \frac{(2k-3)!(k-1)!p^{n(k-1)}}{\tau(\theta^{-1})^2 \alpha^n} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(2\pi i)^{k-1} \Omega_+},
\]

\[
\chi^{2k-3} \left( L_\alpha(\text{Sym}^2(V_f)) \right) = (2k-3)!(k-1)! \left( 1 - \frac{1}{p} + \alpha \left( \frac{p^{-2k+2}}{p \epsilon(p)^2} \right) \right) \times \frac{L(\text{Sym}^2 f, 2k-2)}{(2\pi i)^{k-1} \Omega_+}
\]

where \( \alpha = \pm \epsilon(p)p^{k-1} \).

**Proof.** By definition, \( L_\alpha(\phi^2) \in \mathcal{H}_{k-1,E}(G_\infty) \) and \( L_p(\varepsilon_K \cdot \epsilon) \in \mathcal{H}_{0,E}(G_\infty) \) which implies the first part of the theorem.

Since \( \det(V_f) = \varepsilon_K^{k-1} \) and \( \rho_f \) is odd, we have \( \varepsilon_K^{k-1}(-1) = -1 \). But \( \varepsilon_K(-1) = -1 \) and \( \theta(-1) = 1 \), so \( \chi^{k-1}\theta(-1) = \varepsilon_K\epsilon(-1) \) and we can apply Theorem 3.18 and Lemma 3.15 as follows:

\[
\chi^{2k-3} \left( L_\alpha(\text{Sym}^2(V_f)) \right) = \chi^{2k-3} \left( L_\alpha(\phi^2) \right) \times \chi^{k-2} \left( L_p(\varepsilon_K \cdot \epsilon) \right)
\]

\[
= \frac{(2k-3)!p^{2k-2n}}{\tau(\theta^{-1})^2 \alpha^n} \times \frac{L(\phi^2 \theta, 2k-2)}{(2\pi i)^{k-1} \Omega_+} \times \frac{(k-1)!p^{n(k-1)}}{(2\pi i)^{k-1} \tau(\theta^{-1})} \times L(\varepsilon_K \cdot \epsilon \cdot \theta^{-1}, k-1)
\]

\[
= \frac{(2k-3)!(k-1)!p^{n(k-1)}}{\tau(\theta^{-1})^2 \alpha^n} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(2\pi i)^{k-1} \Omega_+},
\]

where the last equality follows from Corollary 3.4. This gives the first interpolating formula and the second one can be deduced in the same way.

**Lemma 3.21.** Let \( \eta \) be an even character on \( \Delta \), then \( L_{\pm \epsilon(p)p^{k-1}}^0(\text{Sym}^2(V_f)) \neq 0 \).

**Proof.** We have \( L(\text{Sym}^2(V_f), \eta, 2k-2) \neq 0 \) because the critical strip of \( \text{Sym}^2(V_f) \) is \( k-1 < \text{Re}(s) < k \). Therefore, we are done by the interpolating properties given by Theorem 3.20. \( \square \)
3.6. Pollack’s plus and minus splittings. As in [11], we define

\[
\log^+(\gamma) = \prod_{r=0}^{k-3} \prod_{n=1}^{\infty} \frac{\Phi_{2n}(\chi(\gamma)^{-r}\gamma)}{p},
\]
\[
\log^-(\gamma) = \prod_{r=0}^{k-3} \prod_{n=1}^{\infty} \frac{\Phi_{2n-1}(\chi(\gamma)^{-r}\gamma)}{p},
\]

where \( \Phi_m \) denotes the \( p \)-th cyclotomic polynomial. Then, \( \log^\pm(\gamma) \sim \log^{k-1}. \)

**Lemma 3.22.** For an integer \( r \) such that \( 0 \leq r \leq 2k-3 \) and a character \( \theta \) of \( G_n \) which does not factor through \( G_{n-1} \) with \( n \geq 1 \), then

\[
\chi^r \theta(L_{\pm(\gamma)^{p^{k-1}}}(\phi^2)) = (-1)^n \chi^r \theta(L_{-\gamma}(\phi^2)).
\]

**Proof.** This follows from the same calculations as in the proof of Lemma 3.15 thanks to Lemma 3.13(b). \( \square \)

**Corollary 3.23.** We have divisibilities

\[
\log^+(\gamma) | L_{+\gamma}(\phi^2) + L_{-\gamma}(\phi^2);
\]
\[
\log^-(\gamma) | L_{+\gamma}(\phi^2) - L_{-\gamma}(\phi^2).
\]

Similarly,

\[
\log^+(\gamma) | L_{+\gamma}(\phi^2) + L_{-\gamma}(\phi^2);
\]
\[
\log^-(\gamma) | L_{+\gamma}(\phi^2) - L_{-\gamma}(\phi^2).
\]

**Proof.** The first set of divisibilities follows from Lemma 3.22. The second set is then immediate by definition. \( \square \)

This allows us to define the following.

**Definition 3.24.** We define the plus and minus \( p \)-adic \( L \)-functions for \( \text{Sym}^2(V_f) \) by

\[
L^+_p(\text{Sym}^2(V_f)) = \left( L_{+\gamma}(\phi^2) + L_{-\gamma}(\phi^2) \right)/2 \log^+(\gamma);
\]
\[
L^-_p(\text{Sym}^2(V_f)) = \left( L_{+\gamma}(\phi^2) - L_{-\gamma}(\phi^2) \right)/2 \log^-(\gamma).
\]

Similarly, we define the plus and minus \( p \)-adic \( L \)-functions for \( V_2 \) by

\[
L^+_p(\phi^2) = \left( L_{+\gamma}(\phi^2) + L_{-\gamma}(\phi^2) \right)/2 \log^+(\gamma);
\]
\[
L^-_p(\phi^2) = \left( L_{+\gamma}(\phi^2) - L_{-\gamma}(\phi^2) \right)/2 \log^-(\gamma).
\]

It is immediate that we have

\[
L^+_p(\text{Sym}^2(V_f)) = L^+_p(\phi^2) \times \text{Tw}_{-k+1}(L_p(\varepsilon_K \cdot \gamma)).
\]

**Theorem 3.25.** Both \( L^+_p(\text{Sym}^2(V_f)) \) are elements of \( \Lambda_E(G_{\infty}) \) and admit the following interpolating properties:

(a) If \( n \) is even, then

\[
\chi^{2k-n}(L^+_p(\text{Sym}^2(V_f))) = \frac{(2k-3)! (k-1)! p^{2n(k-1)}}{\log^+(\chi^{2k-n}(\gamma)) \tau(\gamma)^{n}} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(2\pi i)^{k-1} \Omega^+},
\]
\[
\chi^{2k-n}(L^-_p(\text{Sym}^2(V_f))) = \frac{(2k-3)! (k-1)! (1-p^{-1})}{\log^+(\chi^{2k-n}(\gamma))} \times \frac{L(\text{Sym}^2 f, 2k-2)}{(2\pi i)^{k-1} \Omega^+};
\]
(b) If \( n \) is odd, then
\[
\chi^{2k-3} \theta \left( L^+_{p}(\text{Sym}^2(V_f)) \right) = \frac{(2k-3)!(k-1)!p^{2n(k-1)}}{\log (-\chi^{2k-3}(\tau)) \tau(\theta^{-1})^2} \frac{\Delta_p(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(2\pi i)^{k-1} \Omega^+},
\]
\[
\chi^{2k-3} \left( L^-_{p}(\text{Sym}^2(V_f)) \right) = \frac{(2k-3)!(k-1)!}{\log (-\chi^{2k-3}(\tau))} \frac{(\epsilon(p)p^{-k+1} - \epsilon(p)^{-1}p^{k-2})}{(2\pi i)^{k-1} \Omega^+} \frac{\Delta_p(\text{Sym}^2 f, 2k-2)}{L(\text{Sym}^2 f, 2k-2)}.
\]
Moreover, \( L^\pm_p(\text{Sym}^2(V_f)) \) are uniquely determined by (a) and (b) respectively.

**Proof.** By the first part of Theorem 3.20, \( L^\pm_{\epsilon(p)p^{k-1}}(\text{Sym}^2(V_f)) \) are both elements of \( \mathcal{H}_{k-1,E}(G_\infty) \). But \( \log^\pm(\gamma) \sim \log^k \), so the quotients above are in \( \mathcal{H}_{0,E}(G_\infty) = \Lambda_E(G_\infty) \).

The interpolating formulae in (a) and (b) follow from those given in Theorem 3.20.

Finally, since \( L^\pm_p(\text{Sym}^2(V_f)) \in \Lambda_E(G_\infty) \), they are uniquely determined by their values at an infinite number of characters, hence the last part of the theorem. \( \square \)

**Lemma 3.26.** Let \( \eta \) be an even character on \( \Delta \), then \( L^\pm_{p,\eta}(\text{Sym}^2(V_f)) \not= 0 \).

**Proof.** The same as the proof of Lemma 3.21. \( \square \)

**Remark 3.27.** Analogues of Theorem 3.25 and Lemma 3.26 for \( L^p_\phi(\phi^2) \) can be deduced in the same way.

**Remark 3.28.** A conjectural generalisation of Pollack’s plus and minus splittings of \( \ell \)-adic \( L \)-functions for motives has been formulated in [2]. Theorem 3.25 gives an affirmative answer to Conjecture 2 of op. cit. for the special case when the motive corresponds to the symmetric square of a CM modular form.

4. Selmer groups

In this section, we define the plus and minus \( p \)-Selmer groups for \( \text{Sym}^2(V_f) \) and relate them to the \( p \)-adic \( L \)-functions \( L^\pm_{p}(\text{Sym}^2(V_f)) \) defined above. By the decomposition given by Proposition 3.3, we only need to define their counterparts for \( V_2 = \tilde{V}_{f'} \), because the Selmer group of \( V_1 \) is relatively well-understood. The \( G_\mathbb{Q} \)-representation \( V_2 \) behaves in exactly the same way as \( V_f \) where \( f' \) is some CM modular form of weight \( 2k-1 \), so many of the results on \( V_2 \) below can be proved using the arguments given in [6]. Therefore, we only outline the proofs without giving all the details here.

4.1. Coleman maps and Selmer groups. As in [6, 7], we define plus and minus Selmer groups using the kernels of some Coleman maps.

**Proposition 4.1.** If \( z \in \mathbb{H}_w^1(V_2^+) \), then
\[
\log^+(\gamma) \mid \mathcal{L}_{\theta \otimes \phi}(z),
\]
\[
\log^-(\gamma) \mid \mathcal{L}_{\omega \otimes \phi}(z).
\]

**Proof.** As in [6, Proposition 3.14], this can be proved using Proposition 3.10. \( \square \)

Therefore, as in [6], we may define \( \Lambda_E(G_\infty) \)-homomorphisms
\[
\text{Col}^+: \mathbb{H}_w^1(V_2^+) \rightarrow \Lambda_E(G_\infty)
\]
\[
z \mapsto \frac{1}{2[\phi(\omega) \otimes \phi(\omega), \omega] \log^+(\gamma)} \mathcal{L}_{\phi(\omega) \otimes \phi(\omega)}(z);
\]
\[
\text{Col}^-: \mathbb{H}_w^1(V_2^+) \rightarrow \Lambda_E(G_\infty)
\]
\[
z \mapsto \frac{1}{2[\phi(\omega) \otimes \phi(\omega), \omega] \log^-(\gamma)} \mathcal{L}_{\phi(\omega)}(z).
\]

Then, it is clear by definition that \( \text{Col}^+ \left( \text{Tw}_{2k-2} \left( z(\phi^2) \right) \right) = L^+_p(\phi^{2^2}) \).
We now fix an \( \mathcal{O}_E \)-lattice \( T \) of \( V(\phi) \) which is stable under \( G_\mathbb{Q} \), it then gives rise to natural \( \mathcal{O}_E \)-lattices \( T_f = \text{Ind}_G^H(T) \) and \( \text{Sym}^2 T_f \) in \( V_f = V_\phi \) and \( \text{Sym}^2(V_f) \) respectively, both of which are again stable under \( G_\mathbb{Q} \). As \( p \neq 2 \), we have
\[
\text{Sym}^2 T_f \cong T_1 \oplus T_2 \quad \text{and} \quad \text{Sym}^2 V_f/T_f \cong V_1/T_1 \oplus V_2/T_2
\]
for some \( \mathcal{O}_E \)-lattice \( T_i \) inside \( V_i \) for \( i = 1, 2 \).

Write \( H^1_\mathbb{Z}(\mathbb{Q}_p, T^2_{\mathbb{Z}}) \) for the projection of \( \text{ker}(\text{Col}^\pm) \) into \( H^1(\mathbb{Q}_p, T^2_{\mathbb{Z}}) \) and define \( H^1(\mathbb{Q}_p, V_2/T_2(1)) \) to be the exact annihilator of \( H^1_\mathbb{Z}(\mathbb{Q}_p, T^2_{\mathbb{Z}}) \) under the Pontryagin duality
\[
H^1(\mathbb{Q}_p, T^2_{\mathbb{Z}}) \times H^1(\mathbb{Q}_p, V_2/T_2(1)) \to \mathbb{Q}_p/\mathbb{Z}_p.
\]

Let \( F \) be a number field. Then the \( p \)-Selmer group of \( \text{Sym}^2 T_f(1) \) decomposes into those of \( T_1(1) \) and \( T_2(1) \):
\[
\text{Sel}_p(\text{Sym}^2 T_f(1)/F) = \text{Sel}_p(T_1(1)/F) \oplus \text{Sel}_p(T_2(1)/F).
\]

We define the plus/minus Selmer groups over \( k_n = \mathbb{Q}(\mu_{p^n}) \) by
\[
\text{Sel}_p^\pm(T_2(1)/k_n) = \text{ker} \left( \text{Sel}_p(T_2(1)/k_n) \to \frac{H^1(\mathbb{Q}_p, V_2/T_2(1))}{H^1(\mathbb{Q}_p, V_2/T_2(1))^{\pm}} \right),
\]
\[
\text{Sel}_p^\pm(\text{Sym}^2 T_f(1)/k_n) = \text{Sel}_p(T_1(1)/k_n) \oplus \text{Sel}_p^\pm(T_2(1)/k_n)
\]
and let
\[
\text{Sel}_p^\pm(T_2(1)/k_\infty) = \lim_{\to} \text{Sel}_p^\pm(T_2(1)/k_n) \quad \text{and} \quad \text{Sel}_p^\pm(\text{Sym}^2 T_f(1)/k_\infty) = \lim_{\to} \text{Sel}_p^\pm(\text{Sym}^2 T_f(1)/k_n).
\]

4.2. Description of the kernels. In this section, we give a more explicit description of the groups \( H^1_\mathbb{Z}(\mathbb{Q}_p, V_2/T_2(1))^{\pm} \) under the following additional assumption.

**Hypothesis 4.2.** Either \( p \equiv \pm 1 \pmod k \) or \( \epsilon \neq 1 \).

In [6, §4], one of the key ingredients to give an explicit description of \( H^1_\mathbb{Z}(\mathbb{Q}_p, V_f/T_f(1))^{\pm} \) is the fact that \( (V_f/T_f(j))^{G_{\mathbb{Q}_p,n}} = 0 \) under some appropriate assumptions. We show below that we get an analogue of such description under Hypothesis 4.2.

**Lemma 4.3.** If Hypothesis 4.2 holds, then \( (V_2/T_2(j))^{G_{\mathbb{Q}_p,n}} = 0 \) for all \( j \in \mathbb{Z} \) and \( n \in \mathbb{Z}_{\geq 0} \).

**Proof.** Let \( p \nmid N \) be a prime which is inert in \( K \). Then, by the second half of the proof of Proposition 3.3, we see that the eigenvalues of the \( p \)-Frobenius on \( V_2(j) \) are \( \pm \epsilon(q) \chi(q) q^{k-1} \). Therefore, as in [6, proof of Lemma 4.4], it is enough to show that there exists some \( q \) such that
\[
\pm \epsilon(q) \chi(q) q^{k-1} \equiv 1 \pmod p.
\]
If either \( p \equiv \pm 1 \pmod k \) or \( \epsilon(q) \neq 1 \), we can find such a \( q \) by Dirichlet’s theorem, so we are done. \( \Box \)

**Corollary 4.4.** If Hypothesis 4.2 holds, then the restriction map \( H^1(\mathbb{Q}_p,m,T_2(1)) \to H^1(\mathbb{Q}_p,m,T_2(1)) \) is injective for any integers \( n \geq m \geq 0 \). On identifying the former as a subgroup of the latter, we have
\[
H^1_\mathbb{Z}(\mathbb{Q}_p, V_2/T_2(1))^{\pm} = H^1_\mathbb{Z}(\mathbb{Q}_p, T_2(1))^{\pm} \otimes E/\mathcal{O}_E.
\]
Here
\[
H^1_\mathbb{Z}(\mathbb{Q}_p, T_2(1))^{\pm} = \{ x \in H^1_\mathbb{Z}(\mathbb{Q}_p, T_2(1)) : \text{cor}_{m/m+1}(x) \in H^1_\mathbb{Z}(\mathbb{Q}_p, T_2(1)) \forall m \in S_n^{\pm} \}
\]
where \( \text{cor} \) denotes the corestriction map and
\[
S_n^+ = \{ m \in [0, m-1] : m \text{ even} \},
\]
\[
S_n^- = \{ m \in [0, m-1] : m \text{ odd} \}.
\]

**Proof.** These can be proved in exactly the same way as their counterparts in [6, §4] using Lemma 4.3. \( \Box \)
4.3. Main conjectures.

**Theorem 4.5.** Let $\theta$ be a character on $\Delta$ and $r \geq 0$ an integer such that $\chi^{r+1}\theta(-1) = \eta(-1)$. Then $\text{Sel}_p(\mathbb{Z}_p(\eta)(r+1))^\theta$ is $\Lambda_E(\Gamma)$-cotorsion and

$$\text{Char}_{\Lambda_E(\Gamma)} \left( \text{Sel}_p(\mathbb{Z}_p(\eta)(r+1))^{\nu,\theta} \right) = (\text{Tw}_r L_p^\theta(\eta)).$$

**Proof.** For any $\Lambda_E(G_\infty)$-module, $M^\nu(r) = M(-r)^\nu$. If $M$ is a $\Lambda_E(\Gamma)$-torsion module, we have $\text{Char}(M^r) = \text{Tw}_r(\text{Char}(M))$. Therefore, the result is just a rewrite of the Iwasawa main conjecture, as proved by Mazur-Wiles [8].

**Corollary 4.6.** Let $\eta$ be an even character on $\Delta$. Then

$$\text{Char}_{\Lambda_E(\Gamma)} (\text{Sel}_p(T_1(1)/k_\infty)^{\nu,\eta}) = (\text{Tw}_{-k+1} L_p^\eta(\varepsilon_K \cdot \epsilon)).$$

**Proof.** We may apply Theorem 4.5 to $\varepsilon_K \cdot \epsilon$ with $r = k - 1$.

**Proposition 4.7.** Let $\delta = \pm$ and let $\eta$ be a character on $\Delta$ such that $\eta = 1$ if $\delta = -$. Then, $\text{Sel}_p^\pm(T_2(1)/k_\infty)^\theta$ is $\Lambda_E(\Gamma)$-cotorsion and

$$\text{Char}_{\Lambda_E(\Gamma)} \left( \text{Sel}_p^\delta(T_2(1)/k_\infty)^{\nu,\eta} \right) = (L_p^\delta(\eta(\phi^2))).$$

**Proof.** This follows from the same argument as in [12], which has been generalised for CM modular forms in [6, §7]. It relies on the main conjecture for $K$ as proved in [13].

**Theorem 4.8.** Let $\eta$ be character on $\Delta$ as in the statement of Proposition 4.7. Then $\text{Sel}_p^\pm(\text{Sym}^2(V_f)/k_\infty)^\eta$ is $\Lambda_E(\Gamma)$-cotorsion and

$$\text{Char}_{\Lambda_E(\Gamma)} \left( \text{Sel}_p^\pm(\text{Sym}^2(V_f)/k_\infty)^{\nu,\eta} \right) = (L_p^\pm(\text{Sym}^2(V_f))).$$

**Proof.** Recall that

$$\text{Sel}_p^\pm(\text{Sym}^2 T_f(1)/k_\infty) = \text{Sel}_p(T_1(1)/k_\infty) \oplus \text{Sel}_p^\pm(T_2(1)/k_\infty)$$

by definition, so

$$\text{Sel}_p^\pm(\text{Sym}^2 T_f(1)/k_\infty)^{\nu,\eta} = \text{Sel}_p(T_1(1)/k_\infty)^{\nu,\eta} \oplus \text{Sel}_p^\pm(T_2(1)/k_\infty)^{\nu,\eta}.$$ 

But we have

$$L_p^\pm(\text{Sym}^2(V_f)) = L_p^\pm(\phi^2) \times \text{Tw}_{-k+1} (L_p^\eta(\varepsilon_K \cdot \epsilon))$$

by (9). Therefore, the theorem follows from Corollary 4.6 and Proposition 4.7 because

$$\text{Char}(M_1 \oplus M_2) = \text{Char}(M_1) \text{Char}(M_2)$$

for any torsion modules $M_1$ and $M_2$.

\section{Appendix}

In this section, we fix an integer $m \geq 2$. We prove an analogue of Proposition 3.3.

**Proposition 5.1.** If $m$ is even, we have a decomposition of $G_\mathbb{Q}$-representations

$$\text{Sym}^m V_f \cong \bigoplus_{i=0}^{m/2-1} \left( \tilde{V}_{\phi^{m-2i}} \otimes (\varepsilon_K \det \rho_f)^{i} \right) \oplus (\varepsilon_K \det \rho_f)^{m/2}. $$

If $m$ is odd, then

$$\text{Sym}^m V_f \cong \bigoplus_{i=0}^{(m-1)/2} \left( \tilde{V}_{\phi^{m-2i}} \otimes (\varepsilon_K \det \rho_f)^{i} \right).$$
Proof. We only give the proof for the case when \( m \) is even since the other case can be proved in a similar way. Let \( x, y \) be the basis of \( V_f \) given as in §3.2. For an integer \( r \) such that \( 0 \leq r \leq m \), we write \( x_r \) for the element in \( V_f^{\otimes m} \) given by

\[
\sum a_1 \otimes a_2 \otimes \cdots \otimes a_m
\]

where the sum runs over \( a_i \in \{ x, y \} \) with \( \# \{ i : a_i = x \} = r \). Then, \( x_0, \ldots, x_m \) give a basis of \( \text{Sym}^m V_f \).

If \( \sigma \in G_K \), we have

\[
\sigma(x_r) = \tilde{\phi}^r(\sigma)\tilde{\phi}^{m-r}(\sigma) x_r
\]

by (2). If \( \sigma = \imath \sigma' \) with \( \sigma' \in G_K \), then

\[
\sigma(x_r) = \tilde{\phi}^r(\sigma')\tilde{\phi}^{m-r}(\imath \sigma') x_{m-r}
\]

by (3). Therefore, \( x_r \) and \( x_{m-r} \) generate a subrepresentation of \( \text{Sym}^m V_f \), which we denote by \( \rho_r : G_K \to \text{GL}(V_r) \) where \( 0 \leq r \leq m/2 \). Note that \( V_r \) is 2-dimensional if \( r < m/2 \) and \( V_{m/2} \) is 1-dimensional. We have a decomposition

\[
\text{Sym}^m V_f \cong \bigoplus_{r=0}^{m/2} V_r.
\]

For \( r < m/2 \), the matrix of \( \sigma \in G_K \) respect to the basis \( x_{m-r}, x_r \) is

\[
\begin{pmatrix}
\tilde{\phi}^m(\sigma) & 0 \\
0 & \tilde{\phi}^r(\sigma)\tilde{\phi}^{m-r}(\sigma) \\
\end{pmatrix}
\]

whereas that of \( \sigma = \imath \sigma' \) with \( \sigma' \in G_K \) is given by

\[
\begin{pmatrix}
0 & \tilde{\phi}^m(\sigma') & 0 \\
\tilde{\phi}^r(\sigma')\tilde{\phi}^{m-r}(\imath \sigma') & 0 \\
0 & \tilde{\phi}^r(\imath \sigma') & 0 \\
\end{pmatrix}
\]

Therefore, we see that \( \rho_r \cong \text{Ind}_K^{G_K}(V(\phi^{m-2r})) \cdot (\varepsilon_K \det \rho_f)^r \) by Lemma 3.2.

Finally, for \( r = m/2 \), we have

\[
\sigma(x_{m/2}) = \begin{cases} 
\tilde{\phi}^{m/2}(\sigma) x_{m/2} & \text{if } \sigma \in G_K \\
\tilde{\phi}^{m/2}(\imath \sigma')x_{m/2} & \text{if } \sigma = \imath \sigma' \text{ where } \sigma' \in G_K.
\end{cases}
\]

Hence, \( V_{m/2} = (\varepsilon_K \det \rho_f)^{m/2} \) again by Lemma 3.2. This finishes the proof. \( \square \)

**Corollary 5.2.** The complex \( L \)-function admits a factorisation

\[
L(\text{Sym}^m f, s) = \begin{cases} 
\prod_{i=0}^{m/2-1} L \left( \phi^{m-2i}, (\varepsilon_K \varepsilon)^s, s - i(k - 1) \right) & \text{if } m \text{ is even,} \\
\prod_{i=0}^{(m-1)/2} L \left( \phi^{m-2i}, (\varepsilon_K \varepsilon)^s, s - i(k - 1) \right) & \text{otherwise.}
\end{cases}
\]

**Proof.** This can be proved in the same way as Corollary 3.4. \( \square \)

**Remark 5.3.** For \( 0 \leq i \leq \lfloor (m - 1)/2 \rfloor \), we may obtain a \( p \)-adic \( L \)-function that interpolates the \( L \)-values of \( \phi^{m-2i} \) at \( (m-2i)(k-1) \) using Proposition 3.1. However, when \( m > 2 \), their product does not interpolate the \( L \)-values of \( \text{Sym}^m f \). We would need \( p \)-adic \( L \)-functions that interpolate the \( L \)-values of \( \phi^{m-2i} \) at \( (m-i)(k-1) \) instead.

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