CRITICAL SLOPE P-ADIC L-FUNCTIONS OF CM MODULAR FORMS

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Abstract. For ordinary modular forms, there are two constructions of a p-adic L-function attached to the non-unit root of the Hecke polynomial, which are conjectured but not known to coincide. We prove this conjecture for modular forms of CM type, by calculating the the critical-slope L-function arising from Kato's Euler system and comparing this with results of Bellaïche on the critical-slope L-function defined using overconvergent modular symbols.

1. Setup

1.1. **Introduction.** Let f be a cuspidal new modular eigenform of weight ≥ 2, and p a prime not dividing the level of f. It has long been known that if α is any root of the Hecke polynomial of f at p such that $v_p(\alpha) < k-1$, then there is a p-adic L-function $L_{p,\alpha}(f)$ interpolating the critical L-values of f and its twists by Dirichlet characters of p-power conductor; see [MSD74, AV75, Viš76].

If f is *non-ordinary* (the Hecke eigenvalue of f at p has valuation > 0) then both roots of the Hecke polynomial satisfy this condition, but if f is ordinary, then there is one root with valuation $k-1$ ("critical slope"), to which the classical modular symbol constructions do not apply. Two approaches exist to rectify this injustice to the ordinary forms by constructing a critical-slope p-adic L-function. Firstly, there is an approach using p-adic modular symbols [PS11, PS09, Bel11a]. Secondly, there is an approach using Kato's Euler system [Kat04] and Perrin-Riou's p-adic regulator map [PR95] (cf. [Col04, Remarque 9.4]). Although it is natural to conjecture that the objects arising from these two constructions coincide (cf. [PS09, Remark 9.7]), and the results of [LZ11b] are strong evidence for this conjecture, prior to the present work this was not known in a single example.

In this paper, we show that the two critical-slope L-functions coincide for modular forms of CM type. In this case, Bellaïche has shown [Bel11b] that the "modular symbol" critical-slope p-adic L-function is related to the Katz p-adic L-function for the corresponding imaginary quadratic field. We show here that the same relation holds for the Kato critical slope p-adic L-function, by comparing Kato's Euler system with another Euler system: that arising from elliptic units. Using the results of [Yag82] and [dS87] relating elliptic units to Katz's L-function, we obtain a formula (Theorem 3.2) for the Kato L-function, which coincides with Bellaïche's formula for its modular symbol counterpart (up to a scalar factor corresponding to the choice of periods). This establishes the equality of the two critical-slope p-adic L-functions for ordinary eigenforms of CM type (Theorem 3.4).

1.2. **Notation.** Let K be a finite extension of either $\mathbb{Q}$ or $\mathbb{Q}_p$, where p is an odd prime. We write $K_\infty = K(\mu_{p^\infty})$, $\overline{K}$ for an algebraic closure of K and K^{ab} for the maximal abelian extension of K in $\overline{K}$. A p-adic representation of the absolute

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Galois group $\mathrm{Gal}(\overline{K}/K)$ is a finite-dimensional $\mathbb{Q}_p$-vector space with a continuous linear action of $\mathrm{Gal}(\overline{K}/K)$.

A Galois extension L of K will be called a p-adic Lie extension if $G = \mathrm{Gal}(L/K)$ is a compact p-adic Lie group of finite dimension. In this case, we denote by $\Lambda(G)$ its Iwasawa algebra; it is defined to be the completed group ring

$$\Lambda(G) = \varprojlim \mathbb{Z}_p[G/U],$$

where U runs over all open normal subgroups of G. We write $Q(G)$ for the total quotient ring of $\Lambda(G)$. If R is a p-adically complete $\mathbb{Z}_p$-algebra, we shall write $\Lambda_R(G)$ for $R \widehat{\otimes} \Lambda(G)$, the Iwasawa algebra with coefficients in R.

If L is a complete discretely valued subfield of $\mathbb{C}_p$, we write $\mathcal{H}_L(G)$ for the algebra of L-valued distributions on G (the continuous dual of the space of locally L-analytic functions). This naturally contains $\Lambda_L(G)$ as a subalgebra. When G is the cyclotomic Galois group Γ (isomorphic to $\mathbb{Z}_p^\times$), and $i \in \mathbb{Z}$, we shall write ℓ_i for the element $\frac{\log([\gamma])}{\log \chi(\gamma)} - i$ of $\mathcal{H}_{\mathbb{Q}_p}(\Gamma)$ (where γ is any element of Γ of infinite order).

Assume now that K is a number field, and let S be a finite set of places of K (which we shall always assume to contain the infinite places). Let K^S be the maximal extension of K which is unramified outside S, and let V be a p-adic representation of $\mathrm{Gal}(K^S/K)$. For an extension L of K contained in K^S, write $H^1_S(L, V)$ for the Galois cohomology group $H^1(\mathrm{Gal}(K^S/L), V)$. Let T be a $\mathrm{Gal}(\overline{K}/K)$-stable lattice in V. If $L \subset K^S$ is a p-adic Lie extension of K, define

$$H^1_{\mathrm{Iw},S}(L, T) = \varprojlim H^1_S(L_n, T),$$

where L_n is a sequence of finite Galois extensions of K such that $L = \bigcup_n L_n$ and the inverse limit is taken with respect to the corestriction maps. Note that $H^1_{\mathrm{Iw},S}(L, T)$ is equipped with a continuous action of $G = \mathrm{Gal}(L/K)$, which extends to an action of $\Lambda(G)$. We also define $H^1_{\mathrm{Iw},S}(L, V) = H^1_{\mathrm{Iw},S}(L, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, which is independent of the choice of lattice T.

Similarly, let F be a finite extension of $\mathbb{Q}_p$, V a p-adic representation of $\mathrm{Gal}(\overline{F}/F)$ and T a $\mathrm{Gal}(\overline{F}/F)$-invariant lattice in V. For a p-adic Lie extension L of F such that $L = \bigcup L_n$ with L_n/F finite Galois, define

$$H^1_{\mathrm{Iw}}(L, T) = \varprojlim H^1(L_n, T) \qquad \text{and} \qquad H^1_{\mathrm{Iw}}(L, V) = H^1_{\mathrm{Iw}}(L, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

For a finite extension K of $\mathbb{Q}$, denote by $\mathbb{A}_K$ the ring of adèles of K. If $\mathfrak{f}$ is an integral ideal of K, write $K(\mathfrak{f})$ for the ray class field modulo $\mathfrak{f}$. Let $K(\mathfrak{f}p^\infty) = \bigcup_n K(\mathfrak{f}p^n)$, and define the Galois group $G_{\mathfrak{f}p^\infty} = \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)$.

1.3. **Grössencharacters.** Let K be an imaginary quadratic field. We fix an embedding $K \hookrightarrow \mathbb{C}$. An algebraic Grössencharacter of K of infinity-type (m, n) is a continuous homomorphism $\psi : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ whose restriction to $\mathbb{C}^\times$ is given by $z \mapsto z^m \bar{z}^n$.

Let θ be the Artin map $\widehat{K}^\times / K^\times \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$. We choose the normalizations such that

$$\theta(\varpi_{\mathfrak{q}}) = [\mathfrak{q}]^{-1} \bmod I_{\mathfrak{q}},$$

where $\varpi_{\mathfrak{q}}$ is a uniformizer at the prime $\mathfrak{q}$, $I_{\mathfrak{q}}$ is the inertia group and $[\mathfrak{q}]$ is the arithmetic Frobenius element at $\mathfrak{q}$. Then we have the following well-known result:

Theorem 1.1 (Weil, [Wei56]). *Let ψ be an algebraic Grössencharacter of K, and let L be the finite extension of $\mathbb{Q}$ inside $\mathbb{C}$ generated by $\psi(\widehat{K}^\times)$. Then for any prime λ of L, there is a (clearly unique) continuous character*

$$\psi_\lambda : \mathrm{Gal}(\overline{K}/K) \longrightarrow L_\lambda^\times$$

with the property that

$$\psi_\lambda \circ \theta = \psi|_{\widehat{K}^\times}.$$

The character ψ_λ is unramified outside the primes dividing $\ell\mathfrak{f}$, where ℓ is the prime of $\mathbb{Q}$ below λ and $\mathfrak{f}$ is the conductor of ψ.

The choice of normalization for the Artin map implies that

$$\psi_\lambda([\mathfrak{a}]) = \psi(\mathfrak{a})^{-1}$$

for each $\mathfrak{a}$ coprime to $\ell\mathfrak{f}$. With these conventions, the Hodge–Tate weights[1] of ψ_λ are given as follows. Let λ be a prime of L, and μ a *split* prime of K, which lie above the same prime of $L \cap K$. Then the decomposition groups of μ and $\overline{\mu}$ in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ are each isomorphic to $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p)$, and the Hodge–Tate weight of ψ_λ is m at μ and n at $\overline{\mu}$.

2. Comparison of Euler systems

2.1. **Elliptic units.** As above, let K be an imaginary quadratic field, with a fixed choice of embedding $K \hookrightarrow \mathbb{C}$. We shall fix, for the remainder of this paper, an embedding $\overline{K} \hookrightarrow \mathbb{C}$ compatible with this choice. In particular, for each integral ideal $\mathfrak{f}$, we regard the ray class field $K(\mathfrak{f})$ as a subfield of $\mathbb{C}$, and we write $K(\mathfrak{f})^+$ for its real subfield[2].

Definition 2.1. *If L is a subfield of $\mathbb{C}$, a* CM-pair *of modulus $\mathfrak{f}$ over L is a pair (E, α) consisting of an elliptic curve E/L and a point $\alpha \in E(L)_{\mathrm{tors}}$, such that*

- *there is an isomorphism $\mathrm{End}_{KL}(E) \cong \mathcal{O}_K$, such that the resulting action of $\mathrm{End}_{KL}(E)$ on $\mathrm{coLie}(E/KL) \cong KL$ is the natural action of K;*
- *the annihilator of α in $\mathcal{O}_K$ is exactly $\mathfrak{f}$;*
- *there is an isomorphism $E(\mathbb{C}) \longrightarrow \mathbb{C}/\mathfrak{f}$ mapping α to 1.*

Note that we do not assume that $L \supseteq K$ here, hence the slightly convoluted statement of the first condition.

Theorem 2.2. *Let $\mathfrak{f}$ be such that $\mathcal{O}_K^\times \cap (1+\mathfrak{f}) = \{1\}$, $\bar{\mathfrak{f}} = \mathfrak{f}$, and the smallest integer in $\mathfrak{f}$ is ≥ 5. Then there exists a CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})^+$, and for any field L containing $K(\mathfrak{f})^+$, this CM-pair is the unique CM-pair of modulus $\mathfrak{f}$ over L up to unique isomorphism.*

Proof. Consider the canonical CM-pair $(\mathbb{C}/\mathfrak{f}, 1)$ over $\mathbb{C}$. This corresponds to a point $P_{\mathfrak{f}}$ on the modular curve $Y_1(N)(\mathbb{C})$, where N is the smallest integer in $\mathfrak{f}$.

Since $N \ge 5$ by assumption, the curve $Y_1(N)$ has a canonical model over $\mathbb{Q}$ such that $Y_1(N)(L)$ parametrises elliptic curves over L with a point of order N for each $L \subseteq \mathbb{C}$. Our claim is then precisely that $P_{\mathfrak{f}} \in Y_1(N)(K(\mathfrak{f})^+)$.

It is clear that $P_{\mathfrak{f}} \in Y_1(N)(\mathbb{R})$, since there is a canonical isomorphism from $\mathbb{C}/\mathfrak{f}$ to the elliptic curve $E_{\mathbb{R}} = \{y^2 = 4x^3 - g_2 x - g_3\}$ where g_2 and g_3 are the usual weight 4 and 6 Eisenstein series, given by $z \mapsto (\wp(z, \mathfrak{f}), \wp'(z, \mathfrak{f}))$. Since $\mathfrak{f} = \bar{\mathfrak{f}}$, the coefficients g_2 and g_3 are real, so $E_{\mathbb{R}}$ is indeed defined over $\mathbb{R}$; and as $\overline{\wp(z, \Lambda)} = \wp(\bar{z}, \bar{\Lambda})$, this uniformization maps $1 \in \mathbb{C}/\mathfrak{f}$ to a real point of $E_{\mathbb{R}}$. Hence $P_{\mathfrak{f}} \in Y_1(N)(\mathbb{R})$.

On the other hand, it is well known that there exists a CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})$ (whether or not $\bar{\mathfrak{f}} = \mathfrak{f}$), so $P_{\mathfrak{f}} \in Y_1(N)(K(\mathfrak{f}))$. Hence $P_{\mathfrak{f}} \in Y_1(N)(K(\mathfrak{f})^+)$. ☐

Remark 2.3. *It follows from this construction that the canonical CM pair (E, α) over $K(\mathfrak{f})^+$ becomes isomorphic over $\mathbb{R}$ to $(E_{\mathbb{R}}, \text{image of } 1 \in \mathbb{C})$. So the complex conjugation automorphism of $E(\mathbb{C})$ arising from this $K(\mathfrak{f})^+$-model corresponds to the natural complex conjugation on $\mathbb{C}/\mathfrak{f}$.*

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[1] We adopt the convention that the cyclotomic character has Hodge–Tate weight $+1$; this is, of course, the Galois character attached to the norm map $\mathbb{A}_K^\times \longrightarrow \mathbb{R}^\times$, which has infinity-type $(1, 1)$.

[2] We stress that $K(\mathfrak{f})$ is not a CM field in general, so the definition of $K(\mathfrak{f})^+$ depends on the choice of embedding, and in particular $K(\mathfrak{f})^+$ is not a totally real field.

We recall the theory of elliptic units, as described in [Kat04, §15.5-6].

Theorem 2.4. *For each pair* $(\mathfrak{f}, \mathfrak{a})$ *of ideals of* K *such that* $\mathcal{O}_K^\times \cap (1+\mathfrak{f}) = \{1\}$ *and* $\mathfrak{a}$ *is coprime to* $6\mathfrak{f}$*, there is a canonical element*

$$ {}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}} \in K(\mathfrak{f})^\times, $$

the elliptic unit *of modulus* $\mathfrak{f}$ *and twist* $\mathfrak{a}$*. If* $\mathfrak{f}$ *has at least two prime factors,* ${}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}} \in \mathcal{O}_{K(\mathfrak{f})}^\times$*; and for any two ideals* $\mathfrak{a}, \mathfrak{b}$ *coprime to* $6\mathfrak{f}$*, we have*

$$ (N(\mathfrak{b}) - [\mathfrak{b}]) \cdot {}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}} = (N(\mathfrak{a}) - [\mathfrak{a}]) \cdot {}_{\mathfrak{b}}\mathbf{e}_{\mathfrak{f}}, $$

where $[\mathfrak{a}] = \left(\frac{\mathfrak{a}}{K(\mathfrak{f})/K}\right) \in \operatorname{Gal}(K(\mathfrak{f})/K)$ *is the arithmetic Frobenius element at* $\mathfrak{a}$*.*

Vital for our purposes is the following complex conjugation symmetry of the elliptic units:

Proposition 2.5. *If* $\mathfrak{f}$ *satisfies the hypotheses of Theorem 2.2, then we have*

$$ \overline{{}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}}} = {}_{\bar{\mathfrak{a}}}\mathbf{e}_{\mathfrak{f}}. $$

Proof. This follows from the construction of the elliptic units. We have

$$ {}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}} = {}_{\mathfrak{a}}\theta_E(\alpha)^{-1} $$

where (E, α) is the canonical CM pair over $K(\mathfrak{f})$, and ${}_{\mathfrak{a}}\theta_E$ is the element of the function field of E constructed in [Kat04, §15.4].

By Theorem 2.2, E admits a model over $K(\mathfrak{f})^+$, and it is clear that if ι is the nontrivial element of $\operatorname{Gal}(K(\mathfrak{f})/K(\mathfrak{f})^+)$ arising from complex conjugation, we have $\iota({}_{\mathfrak{a}}E) = {}_{\bar{\mathfrak{a}}}E$ and hence (by the uniqueness of ${}_{\mathfrak{a}}\theta_E$) we have $({}_{\mathfrak{a}}\theta_E)^\iota = {}_{\bar{\mathfrak{a}}}\theta_E$. Since $\alpha \in E(K(\mathfrak{f})^+)$, we deduce that

$$ \overline{{}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}}} = ({}_{\mathfrak{a}}\theta_E)^\iota(\alpha)^{-1} = {}_{\bar{\mathfrak{a}}}\theta_E(\alpha)^{-1} = {}_{\bar{\mathfrak{a}}}\mathbf{e}_{\mathfrak{f}} $$

as required. □

Remark 2.6. *Modulo differing choices of conventions, this is the formula labelled "Transport of Structure" in §2.5 of* [Gro80].

2.2. **Elliptic units in Iwasawa cohomology.** Let p be a rational prime which splits in K. For fixed $\mathfrak{f}$ (which we shall assume prime to p), the ideal $\mathfrak{g} = \mathfrak{f}p^n$ satisfies the condition $\mathcal{O}_K^\times \cap (1+\mathfrak{g}) = \{1\}$ for all $n \gg 0$, so if $(\mathfrak{a}, 6p\mathfrak{f}) = 1$ we may define the elements ${}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^n}$. These are *norm-compatible* (c.f. [Kat04, §15.5]), and we may extend their definition to all $n \ge 0$ using the norm maps.

Note 2.7. *Since* $\mathfrak{f}p^n$ *has at least two prime factors for* $n \ge 1$*, we have* ${}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^n} \in \mathcal{O}_{K(\mathfrak{f}p^n)}^\times$*.*

Let S be a set of places of K containing the infinite places and the primes above p. Then we have the Kummer maps

$$ \kappa_L : \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_{L,S}^\times \xrightarrow{\cong} H^1_S(L, \mathbb{Z}_p(1)). $$

Since the sequence of elements ${}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^\infty} = ({}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^n})_{n\ge 0}$ is a norm-compatible sequence of units, their images under the Kummer maps are corestriction-compatible, so we obtain an element

$$ {}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^\infty} \in H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1)) = \varprojlim_n H^1_S(K(\mathfrak{f}p^n), \mathbb{Z}_p(1)). $$

Theorem 2.8. *If* $\mathfrak{f}$ *is Galois-stable, then we have*

$$ \iota_*\left({}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^\infty}\right) = {}_{\bar{\mathfrak{a}}}\mathbf{e}_{\mathfrak{f}p^\infty}, $$

where ι_* *is the involution of* $H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1))$ *induced by complex conjugation.*

Proof. Immediate from Proposition 2.5, since $\mathfrak{f}p^n$ satisfies the conditions of Theorem 2.2 for all $n \gg 0$. □

Definition 2.9. *We also define the element*

$$\mathbf{e}_{\mathfrak{f}p^\infty} = (N(\mathfrak{a}) - [\mathfrak{a}])^{-1} \cdot {}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^\infty} \in Q(G_{\mathfrak{f}p^\infty}) \otimes_{\Lambda(G_{\mathfrak{f}p^\infty})} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1)),$$

where $\Lambda(G_{\mathfrak{f}p^\infty})$ is the Iwasawa algebra of $G_{\mathfrak{f}p^\infty} = \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)$ and $Q(G_{\mathfrak{f}p^\infty})$ its total ring of quotients.

Note 2.10. *The element $\mathbf{e}_{\mathfrak{f}p^\infty}$ is independent of the choice of $\mathfrak{a}$.*

Corollary 2.11. *We have $\iota_*(\mathbf{e}_{\mathfrak{f}p^\infty}) = \mathbf{e}_{\bar{\mathfrak{f}}p^\infty}$.*

Proof. The automorphism ι_* of $H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1))$ is $\Lambda(G_{\mathfrak{f}p^\infty})$-semilinear, with the action of ι on $G_{\mathfrak{f}p^\infty}$ being given by conjugation in $\mathrm{Gal}(\overline{K}/\mathbb{Q})$; hence ι_* extends canonically to the tensor product with $Q(G_{\mathfrak{f}p^\infty})$; and since $\iota[\mathfrak{a}]\iota = [\bar{\mathfrak{a}}]$, this finishes the proof by Theorem 2.8. □

Let W be any continuous representation of $G_{\mathfrak{f}p^\infty}$ on a one-dimensional vector space over some finite extension L of $\mathbb{Q}_p$. Then we have an isomorphism

$$(1) \qquad H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} W \xrightarrow{\cong} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), W(1)).$$

Definition 2.12. *For an element $w \in W$, let $\mathbf{e}_{\mathfrak{f}p^\infty}(w)$ be the image of $\mathbf{e}_{\mathfrak{f}p^\infty} \otimes w$ under (1), which is an element of*

$$Q(G_{\mathfrak{f}p^\infty}) \otimes_{\Lambda(G_{\mathfrak{f}p^\infty})} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), W(1)).$$

Define

$$\mathbf{e}_\infty(w) \in Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^1_{\mathrm{Iw},S}(K_\infty, W(1))$$

to be the image of $\mathbf{e}_{\mathfrak{f}p^\infty}(w)$ under the corestriction map

$$H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), W(1)) \longrightarrow H^1_{\mathrm{Iw},S}(K_\infty, W(1)).$$

Lemma 2.13. *If W has no fixed points under $\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K_\infty)$, then we have*

$$\mathbf{e}_\infty(w) \in H^1_{\mathrm{Iw},S}(K_\infty, W(1)).$$

Proof. Let I be the ideal in $\Lambda(\mathfrak{f}p^\infty)$ generated by the elements $(N\mathfrak{a} - [\mathfrak{a}])$ for integral ideals $\mathfrak{a}$ prime to $6\mathfrak{f}$. Suppose $G_{\mathfrak{f}p^\infty}$ acts on W via the character $\tau : G_{\mathfrak{f}p^\infty} \longrightarrow L$. Then we must show that the ideal in $\Lambda(\Gamma)$ generated by the elements

$$\{(N\mathfrak{a} - \tau([\mathfrak{a}])^{-1}[\mathfrak{a}]) : \mathfrak{a} \text{ is an integral ideal coprime to } 6\mathfrak{f}\}$$

contains a power of p. However, if this is not the case, it must consist of elements of $\Lambda(\Gamma)$ which all vanish at some character η of Γ. Then $\chi([\mathfrak{a}])\tau([\mathfrak{a}]) - \eta([\mathfrak{a}])$ vanishes for every $\mathfrak{a}$. By the Chebotarev density theorem, we must have $\tau = \chi^{-1}\eta$, which contradicts the assumption that τ does not factor through Γ. □

We write ιW for the representation of $G_{\mathfrak{f}p^\infty}$ that acts on $\{\iota w : w \in W\}$ via $g \cdot (\iota w) = \iota(\iota g \iota) \cdot w$.

Theorem 2.14. *If W has no fixed points under $\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K_\infty)$, the element*

$$\mathbf{e}_\infty(w) \in H^1_{\mathrm{Iw},S}(K_\infty/K, W(1))$$

satisfies

$$\iota_*(\mathbf{e}_\infty(w)) = \mathbf{e}_\infty(\iota w)$$

where ι_* is induced from the maps

$$H^1_S(K(\mathfrak{f}p^n), W(1)) \longrightarrow H^1_S(K(\bar{\mathfrak{f}}p^n), (\iota W)(1))$$

sending a cocycle τ to the cocycle $g \mapsto \iota\tau(\iota g \iota)$, for each $n \geq 0$.

We split the proof of the theorem into a number of steps.

Definition 2.15. *Let* $\Lambda^\sharp(G_{\mathfrak{f}p^\infty})(1)$ *denote* $\Lambda(G_{\mathfrak{f}p^\infty})(1)$ *endowed with the action of* $\mathrm{Gal}(K^S/K)$ *via the product of the cyclotomic character with the inverse of the canonical character* $\mathrm{Gal}(K^S/K) \twoheadrightarrow G_{\mathfrak{f}p^\infty} \hookrightarrow \Lambda(G_{\mathfrak{f}p^\infty})^\times$*, i.e.* $g.\omega = \chi(g)\bar{g}^{-1}\omega$ *for any* $g \in \mathrm{Gal}(K^S/K)$ *and* $\omega \in \Lambda^\sharp(G)$*. Here,* $\bar{g}$ *denotes the image of* g *in* $G_{\mathfrak{f}p^\infty}$*.*

Lemma 2.16. *We have a commutative diagram*

$$\begin{array}{ccc} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty),\mathbb{Z}_p(1))\otimes_{\mathbb{Z}_p} W & \xrightarrow{\cong} & H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty),W(1)) \\ \Big\downarrow \iota_*\otimes\iota & & \Big\downarrow \iota_* \\ H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty),\mathbb{Z}_p(1))\otimes_{\mathbb{Z}_p} \iota W & \xrightarrow{\cong} & H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty),(\iota W)(1)) \end{array} \tag{2}$$

where the left-hand vertical map is the tensor product of the automorphism ι_* *of* $H^1_{\mathrm{Iw},S}(K_\infty,\mathbb{Z}_p(1))$ *and the canonical map* $\iota : W \longrightarrow \iota W$*, and the right-hand vertical map is as defined in the statement of Theorem 2.14.*

Proof. We will deduce this isomorphism by using an alternative definition of the Iwasawa cohomology which renders the horizontal maps in the diagram easier to handle. By Shapiro's lemma, we have a canonical isomorphism of $\Lambda(G_{\mathfrak{f}p^\infty})$-modules

$$H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty),M(1)) \cong H^1_S(K, M\otimes_{\mathbb{Z}_p}\Lambda^\sharp\big(G_{\mathfrak{f}p^\infty})(1)\big)$$

for any $\mathrm{Gal}(K^S/K)$-module M which is finite-rank over $\mathbb{Z}_p$ or $\mathbb{Q}_p$.

Let τ be the character by which $G_{\mathfrak{f}p^\infty}$ acts on W, and define $\tau_* : \Lambda^\sharp(G)\to\Lambda^\sharp(G)$ to be the map induced by $g \to \tau(g)^{-1}g$. Then the natural twisting map

$$j : H^1_S\big(K,\Lambda^\sharp(G)(1)\big)\otimes W \xrightarrow{\cong} H^1_S\big(K,\Lambda^\sharp(G)(1)\otimes W\big),$$

is explicitly given as follows: if $c : \mathrm{Gal}(K^S/K)\to\Lambda^\sharp(G)(1)$ is a cocycle and $w\in W$, define

$$j(c\otimes w)(g) = \tau_*(c(g))\otimes w.$$

We check that $j(c\otimes w)$ is a cocycle. Let $h, g\in \mathrm{Gal}(K^S/K)$. Then

$$\begin{aligned} j(c\otimes w)(gh) &= \tau_*(c(gh))\otimes w \\ &= \tau_*(g.c(h))\otimes w + \tau_* c(g)\otimes w \\ &= \chi(g)\tau_*(g^{-1}c(h))\otimes w + \tau_* c(g)\otimes w \\ &= \chi(g)\tau(g)\ g^{-1}[\tau_*(c(h))]\otimes w + \tau_*(c(g))\otimes w \\ &= g.[j(c\otimes w)(h)] + j(c\otimes w)(g) \end{aligned}$$

Rewrite the diagram (2) as

$$\begin{array}{ccc} H^1_S(K,\Lambda^\sharp(G)(1))\otimes_{\mathbb{Z}_p} W & \xrightarrow{j_W} & H^1_S(K,\Lambda^\sharp(G)(1)\otimes W) \\ \Big\downarrow \iota_*\otimes\iota & & \Big\downarrow \iota_* \\ H^1_S(K,\Lambda^\sharp(G)(1))\otimes_{\mathbb{Z}_p} \iota W & \xrightarrow{j_{\iota W}} & H^1_S(K,\Lambda^\sharp(G)(1)\otimes \iota W) \end{array} \tag{3}$$

It is then immediate from the description of j that the diagram commutes, which finishes the proof. □

Proof of Theorem 2.14. By Corollary 2.11 and Lemma 2.16, we have

$$\iota_*(\mathbf{e}_{\mathfrak{f}p^\infty}(w)) = \mathbf{e}_{\mathfrak{f}p^\infty}(\iota w).$$

The action of ι_* is clearly compatible with corestriction, so we have a commutative diagram

$$\begin{array}{ccc} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), W(1)) & \longrightarrow & H^1_{\mathrm{Iw},S}(K_\infty, W(1)) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota_*} \\ H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), (\iota W)(1)) & \longrightarrow & H^1_{\mathrm{Iw},S}(K_\infty, \iota W(1)) \end{array}$$

which implies that $\iota_*(\mathbf{e}_\infty(w)) = \mathbf{e}_\infty(\iota w)$, completing the proof. ☐

Lemma 2.17. *Let V be any p-adic representation of $\mathrm{Gal}(K^S/\mathbb{Q})$. Then the restriction map induces an isomorphism*

$$H^1_{\mathrm{Iw},S}(\mathbb{Q}_\infty, V) \longrightarrow H^1_{\mathrm{Iw},S}(K_\infty, V)^{\mathrm{Gal}(K_\infty/\mathbb{Q}_\infty)}.$$

Proof. The restriction map is induced from the restriction maps on finite level, which fit into the exact sequence

$$0 \longrightarrow H^1\big(\mathrm{Gal}(K_n/\mathbb{Q}_n), V^{\mathrm{Gal}(K^S/K_n)}\big) \longrightarrow H^1_S(\mathbb{Q}_n, V)$$
$$\longrightarrow H^1_S(K_n, V)^{\mathrm{Gal}(K_n/\mathbb{Q}_n)} \longrightarrow H^2\big(\mathrm{Gal}(K_n/\mathbb{Q}_n), V^{\mathrm{Gal}(K^S/K_n)}\big).$$

Since $\mathbb{Q}_p$ has characteristic 0, the higher cohomology groups of any $\mathbb{Q}_p$-linear representation of the cyclic group of order 2 are zero. This gives the claim at each finite level, and hence in the inverse limit. ☐

Let α be the unique nontrivial element of $\mathrm{Gal}(K_\infty/\mathbb{Q}_\infty)$.

Lemma 2.18. *We have $\alpha = \delta\iota$, where δ is the unique element of $\mathrm{Gal}(K_\infty/K)$ which acts on $\mathbb{Q}_\infty$ as complex conjugation. In particular, δ is of order 2.*

Corollary 2.19. *If α is the unique nontrivial element of $\mathrm{Gal}(K_\infty/\mathbb{Q}_\infty)$, then for any $w \in W$,*

$$\alpha_*\left(\mathbf{e}_\infty(w)\right) = \delta \cdot \mathbf{e}_\infty(\iota w).$$

Proof. As above, write $\alpha = \delta\iota$. By Lemma 2.17, we have $\iota^* \cdot e_\infty(w) = \mathbf{e}_\infty(\iota w)$. Hence $\alpha_*\left(\mathbf{e}_\infty(w)\right) = \delta \cdot \iota_*\left(\mathbf{e}_\infty(w)\right) = \delta \cdot \mathbf{e}_\infty(\iota w)$. ☐

2.3. **The two-variable L-function of K.** We recall the construction (originally due to Yager [Yag82]) of a two-variable p-adic L-function from the elliptic units.

Let $\mathfrak{p}$ be one of the two primes of K above p. We choose an embedding $\overline{K} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing the $\mathfrak{p}$-adic valuation on K. Then for any finite extension L/K, and any $\mathrm{Gal}(\overline{K}/K)$-module M, we may define

$$Z^1_{\mathfrak{p}}(L, M) = \bigoplus_{\mathfrak{q}|\mathfrak{p}} H^1(L_{\mathfrak{q}}, M) = H^1(K_{\mathfrak{p}}, \mathrm{Ind}_L^K M).$$

which is a $\mathrm{Gal}(L/K)$-module. We also define

$$Z^1_{\mathrm{Iw},\mathfrak{p}}(K(\mathfrak{f}p^\infty), M) = \varprojlim_L Z^1_{\mathfrak{p}}(L, M)$$

where the limit is taken over finite extensions L/K contained in $K(\mathfrak{f}p^\infty)$.

We now recall the theory of two-variable Coleman series, as introduced, under certain additional hypotheses, by Yager [Yag82], and generalized to the semi-local situation here by de Shalit [dS87, §II.4.6]. Let $\zeta = (\zeta_{p^n})_{n\geq 0}$ be a compatible system of p-power roots of unity in $\overline{K}$; and let $\widehat{F}_\infty$ be the completion of $K(\mathfrak{f}\bar{\mathfrak{p}}^\infty)$ with respect to the prime $\mathfrak{P}$ of $\overline{K}$ above $\mathfrak{p}$ induced by our choice of embedding $\overline{K} \hookrightarrow \overline{\mathbb{Q}}_p$, and $\widehat{\mathcal{O}}_\infty$ the ring of integers of $\widehat{F}_\infty$. (Thus $\widehat{\mathcal{O}}_\infty$ is a complete discrete valuation ring with maximal ideal generated by p, and its residue field is a finite extension of the unique $\mathbb{Z}_p$-extension of $\mathbb{F}_p$.)

Proposition 2.20. *There is a unique morphism of $\Lambda(G_{\mathfrak{f}p^\infty})$-modules*

$$\mathrm{Col}^\zeta : Z^1_{\mathrm{Iw},\mathfrak{p}}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1)) \longrightarrow \Lambda_{\widehat{\mathcal{O}}_\infty}(G_{\mathfrak{f}p^\infty})$$

with the following property:

For each finite-order character η of $G_{\mathfrak{f}p^\infty}$ which is not unramified at $\mathfrak{p}$, we have

$$\mathrm{Col}^\zeta(u)(\eta) = \tau(\eta,\zeta)^{-1}\eta(\tilde{\varphi})^n \left(\sum_{\sigma \in G_{\mathfrak{f}p^m}} \eta(\sigma)^{-1} \log_{\mathfrak{P}}(u_m^\sigma) \right).$$

Here $\tilde{\varphi}$ is the unique lifting of the arithmetic Frobenius of $\mathrm{Gal}(K(\mathfrak{f}\bar{\mathfrak{p}}^\infty)/K)$ to $\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K_\infty)$, m is any integer such that η factors through the quotient $G_{\mathfrak{f}p^m} = \mathrm{Gal}(K(\mathfrak{f}p^m)/K)$, $\log_{\mathfrak{P}}$ is the logarithm map

$$\mathcal{O}^\times_{K(\mathfrak{f}p^n),\mathfrak{P}} \longrightarrow K(\mathfrak{f}p^n)_{\mathfrak{P}},$$

and

$$\tau(\eta,\zeta) = \sum_{\sigma \in \mathrm{Gal}(K(\mathfrak{f}\bar{\mathfrak{p}}^\infty)(\mu_{p^n})/K(\mathfrak{f}\bar{\mathfrak{p}}^\infty))} \omega(\sigma)^{-1}\zeta_{p^n}^\sigma,$$

where n is the exact power of $\mathfrak{p}$ dividing the conductor of η.

Definition 2.21. *We let*

$$\mathbb{L}_{\mathfrak{f}p^\infty} = \mathrm{Col}^\zeta(\mathbf{e}_{\mathfrak{f}p^\infty}) \in \widehat{\mathcal{O}}_\infty \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} Q(G_{\mathfrak{f}p^\infty}).$$

Proposition 2.22. *The element $\mathbb{L}_{\mathfrak{f}p^\infty}$ lies in $\Lambda_{\widehat{\mathcal{O}}_\infty}(G_{\mathfrak{f}p^\infty})$, and it coincides with the measure $\mu(\mathfrak{f}\bar{\mathfrak{p}}^\infty)$ in* [dS87, Theorem II.4.14].

Proof. We have $(N\mathfrak{a} - [\mathfrak{a}]) \cdot \mathbb{L}_{\mathfrak{f}p^\infty} \in \Lambda_{\widehat{\mathcal{O}}_\infty}(G_{\mathfrak{f}p^\infty})$ for all $\mathfrak{a}$. Since the ideal generated by $N\mathfrak{a} - [\mathfrak{a}]$ for all integral ideals $\mathfrak{a}$ coprime to $6\mathfrak{f}$ has height 2, this implies that $\mathbb{L}_{\mathfrak{f}p^\infty} \in \Lambda_{\widehat{\mathcal{O}}_\infty}(G_{\mathfrak{f}p^\infty})$ (cf. [dS87, §II.4.12]).

To show that the resulting measure coincides with de Shalit's $\mu(\mathfrak{f}\bar{\mathfrak{p}}^\infty)$, we compare the defining property of the map Col above with [dS87, Theorem II.5.2]. For a finite-order character η of $G_{\mathfrak{f}p^n}$, whose conductor $\mathfrak{g}$ is divisible by $\mathfrak{p}$ and satisfies $\mathcal{O}_K^\times \cap (1+\mathfrak{g}) = \{1\}$, de Shalit shows that

$$\eta(\mu(\mathfrak{f}\bar{\mathfrak{p}}^\infty)) = \frac{-1}{12g} G(\eta) \sum_{\mathfrak{c} \in \mathrm{Cl}(\mathfrak{g})} \eta^{-1}([\mathfrak{c}]) \log \phi_{\mathfrak{g}}(\mathfrak{c}),$$

where g is the smallest rational integer in $\mathfrak{g}$, $\phi_{\mathfrak{g}}(\mathfrak{c})$ is Robert's invariant and the quantity $G(\eta)$ coincides with what we have called $\tau(\eta,\zeta)^{-1}\eta(\tilde{\varphi})^n$. Since

$$(N(\mathfrak{a}) - [\mathfrak{a}])\phi_{\mathfrak{g}}(\mathfrak{c}) = [\mathfrak{c}] \cdot ({}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{g}})^{-12g},$$

this shows that the two measures coincide at every finite-order character, and hence they are equal in $\Lambda_{\widehat{\mathcal{O}}_\infty}(G_{\mathfrak{f}p^\infty})$. □

Note 2.23. *If one identifies $G(\mathfrak{f}p^\infty)$ with the ray class group modulo $\mathfrak{f}p^\infty$ via the Artin map, normalized as in §1.3 above, then this measure coincides with the pull-back of the Katz two-variable L-function of K (cf.* [HT93, §4]*) up to a difference of signs. This remark will be important in the proof of Theorem 3.4 below.*

2.4. **Kato's zeta element.** Let $f = \sum a_n q^n$ be a modular form of CM type, corresponding to a Grössencharacter ψ of K with infinity-type $(1-k, 0)$ where k is the weight of f. It is clear that the coefficient field $F = \mathbb{Q}(a_n : n \geq 1)$ of f is contained in the finite extension L/K contained in $\mathbb{C}$ generated by $\psi(\widehat{K}^\times)$.

Following [Kat04, §6.3], we write $S(f)$ and $V(f)$ for the subspaces of the de Rham and Betti cohomology of the Kuga–Sato variety attached to f. Note that both of these are F-vector spaces, and $S(f)$ is 1-dimensional over F while $V(f)$ is 2-dimensional. For a commutative ring A over F, define $S_A(f) = S(f) \otimes_F A$ and $V_A(f) = V(f) \otimes_F A$. If λ is a place of F above p, we may identify $V_{F_\lambda}(f)$ with the p-adic representation associated to f of Deligne [Del69] and $S_{F_\lambda}(f)$ may be identified with $\mathrm{Fil}^1\, \mathbb{D}_{\mathrm{cris}}(V_{F_\lambda}(f))$.

Definition 2.24. *Let χ be a Dirichlet character of conductor p^n. We define the maps $\theta^\pm_{\chi,f}$ by*

$$\begin{array}{cccc} \theta^\pm_{\chi,f} : & S(f) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_{p^n}) & \longrightarrow & V_{\mathbb{C}}(f)^\pm \\ & x \otimes y & \longmapsto & \sum_{\sigma \in G_n} \chi(\sigma)\sigma(y)\,\mathrm{per}_f(x)^\pm \end{array}$$

where $G_n = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$, $\mathrm{per}_f : S(f) \longrightarrow V_{\mathbb{C}}(f)$ is the period map as defined in [Kat04, §6.3] and $\gamma \mapsto \gamma^\pm$ is the projection from $V_{\mathbb{C}}(f)$ to its (1-dimensional) ± 1-eigenspace for the complex conjugation.

Theorem 2.25 ([Kat04, Theorem 12.5(1)])**.** *We have a L_λ-linear map*

$$\begin{array}{ccc} V_{L_\lambda}(f) & \longrightarrow & H^1_{\mathrm{Iw},S}(\mathbb{Q}_\infty, V_\lambda(f)) \\ \gamma & \longmapsto & \mathbf{z}^{\mathrm{Kato}}_\gamma \end{array}$$

which satisfies the following. Let χ be a Dirichlet character of conductor p^n, $\gamma \in V_L(f)$ and $1 \leq r \leq k-1$, then

$$\theta^\pm_{\chi,f} \circ \exp^*\left(\mathbf{z}^{\mathrm{Kato}}_\gamma \otimes (\zeta_{p^n})^{\otimes(k-r)}\right) = (2\pi i)^{k-r-1} L_{\{p\}}(f^*, \chi, r) \cdot \gamma^\pm$$

where $\pm = (-1)^{k-r-1}\chi(-1)$.

Let $\mathfrak{f}$ be an ideal of $\mathcal{O}_K$ satisfying the conditions in Theorem 2.2 which is contained in the conductor of ψ. Let (E, α) be the canonical CM-pair over $K(\mathfrak{f})$. Following [Kat04, §15.8], we define $V_L(\psi) = H^1(E(\mathbb{C}), \mathbb{Q})^{\otimes(k-1)} \otimes_K L$ and $S(\psi) = H^0(\mathrm{Gal}(K(\mathfrak{f})/K), \mathrm{coLie}(E)^{\otimes(k-1)} \otimes_K L)$, where the action of $\mathrm{Gal}(K(\mathfrak{f})/K)$ on the space $\mathrm{coLie}(E)^{\otimes(k-1)} \otimes_K L$ is as described in *op.cit.*. Both of these are 1-dimensional L-vector spaces. For any commutative ring A over L, we write $V_A(\psi) = V_L(\psi) \otimes_L A$ and $S_A(\psi) = S(\psi) \otimes_L A$. The Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on $V_L(\psi) \otimes_L L_\lambda$ via ψ_λ, and there exists a period map

$$\mathrm{per}_\psi : S(\psi) \longrightarrow V_{\mathbb{C}}(\psi)$$

induced by passing to the $(k-1)$-st tensor power from the comparison isomorphism per_∞ described above.

We now recall Kato's results on the relation between this zeta element and the elliptic units.

Lemma 2.26 ([Kat04, Lemma 15.11])**.** *Fix a choice of isomorphism of L-vector spaces*

$$s : S(\psi) \xrightarrow{\sim} S_L(f).$$

(a) *There exists a unique isomorphism of representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *over* L_λ

$$\widetilde{V_{L_\lambda}(\psi)} \longrightarrow V_{L_\lambda}(f)$$

such that the isomorphism $S_{L_\lambda}(\psi) \longrightarrow S_{L_\lambda}(f)$ induced by the functoriality of $\mathbb{D}_{\mathrm{dR}}$ is compatible with s.

(b) *There exists a unique isomorphism of representations of* $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ *over* L

$$\widetilde{V_L(\psi)} \longrightarrow V_L(f)$$

for which the diagram

$$\begin{array}{ccc} S(\psi) & \xrightarrow{\mathrm{per}_\psi} & \widetilde{V_{\mathbb{C}}(\psi)} \\ \downarrow & & \downarrow \\ S_L(f) & \xrightarrow{\mathrm{per}_f} & V_{\mathbb{C}}(f) \end{array}$$

commutes.

Note that the isomorphism of part (b) implies an isomorphism $V_{L_\lambda}(\psi) \xrightarrow{\cong} V_{L_\lambda}(f)$ on extending scalars to L_λ, but one does not know that this coincides with the isomorphism of part (a), as remarked in [Kat04, §15.11].

Definition 2.27. *We write* $\Phi_{\psi,f}$ *for the canonical map*

$$H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), V_{L_\lambda}(\psi)) \longrightarrow H^1_{\mathrm{Iw},S}(\mathbb{Q}_\infty, V_{L_\lambda}(f))$$

as defined in [Kat04, (15.12.1)].

Concretely, this map can be defined as follows:

$$\begin{aligned} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), V_{L_\lambda}(\psi)) &\longrightarrow H^1_S(K, \Lambda^\sharp(\Gamma) \otimes V_{L_\lambda}(\psi)) \longrightarrow \\ H^1_S(\mathbb{Q}, \operatorname{Ind}_K^{\mathbb{Q}} \left(\Lambda^\sharp(\Gamma) \otimes V_{L_\lambda}(\psi)\right)) &\xrightarrow{\cong} H^1_S(\mathbb{Q}, \Lambda^\sharp(\Gamma) \otimes V_{L_\lambda}(f)). \end{aligned}$$

Theorem 2.28. *Let* $\gamma \in V_L(\psi)$ *and write* γ' *for its image in* $V_L(f)$ *under the map given by Lemma 2.26(b). Then we have*

$$\Phi_{\psi,f}\left(\mathbf{e}_\infty(\gamma) \otimes (\zeta_{p^n})^{\otimes(-1)}\right) = \mathbf{z}^{\mathrm{Kato}}_{\gamma'}.$$

Proof. This is [Kat04, (15.16.1)]; it is immediate from a comparison the interpolating properties of the two zeta elements, since an element of $H^1_{\mathrm{Iw}}(\mathbb{Q}_\infty/\mathbb{Q}, V_{L_\lambda}(f))$ is uniquely determined by its images under the dual exponential maps at each finite level in the tower $\mathbb{Q}_\infty/\mathbb{Q}$. □

Proposition 2.29. *We have a commutative diagram*

$$\begin{array}{ccc} H^1_{\mathrm{Iw},S}(K_\infty, V_{L_\lambda}(\psi)) & \xrightarrow{\Phi_{\psi,f}} & H^1_{\mathrm{Iw},S}(\mathbb{Q}_\infty, V_{L_\lambda}(f)) \\ \downarrow & \swarrow_{\cong} & \\ H^1_{\mathrm{Iw},S}(K_\infty, V_{L_\lambda}(\psi) \oplus \iota V_{L_\lambda}(\psi))^{\alpha=1} & & \end{array}$$

where the left-hand vertical map sends x *to* $x \oplus \delta \cdot \iota_*(x)$*, and the diagonal isomorphism is given by restriction.*

Proof. Clear. □

3. Critical-slope L-functions

Let f be a modular form of CM type, as above, and ψ the corresponding Grössencharacter. We choose a basis γ of $V_L(\psi)$, and let γ' be its image in $V_L(f)$ under the isomorphism of Lemma 2.26(b).

We fix an embedding $\overline{K} \hookrightarrow \overline{\mathbb{Q}}_p$ which induces the λ-adic valuation on L. This gives an embedding $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\overline{K}/\mathbb{Q})$, whose image is contained in the subgroup $\mathrm{Gal}(\overline{K}/K)$. This gives a localization map

$$\mathrm{loc}_p : H^1_{\mathrm{Iw},S}(\mathbb{Q}_\infty, M) \longrightarrow H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, M)$$

for each $\mathrm{Gal}(K^S/\mathbb{Q})$-module M. Moreover, we have a map

$$\mathrm{loc}_{\mathfrak{p}} : H^1_{\mathrm{Iw},S}(K_\infty, M) \longrightarrow H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, M)$$

for each $\mathrm{Gal}(K^S/K)$-module M, and we clearly have $\mathrm{loc}_p = \mathrm{loc}_{\mathfrak{p}} \circ \mathrm{res}_{K/\mathbb{Q}}$.

Via the isomorphism of Lemma 2.26(a), the space $V_{L_\lambda}(f)$ is isomorphic as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to $V_{L_\lambda}(\psi) \oplus \iota\left(V_{L_\lambda}(\psi)\right)$. Note that ι does not normalize the image of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, so the two factors are non-isomorphic; indeed $V_{L_\lambda}(\psi)$ has Hodge–Tate weight $1-k$, while $\iota\left(V_{L_\lambda}(\psi)\right)$ has Hodge–Tate weight 0. Hence we have

$$\mathrm{loc}_p(\mathbf{z}^{\mathrm{Kato}}_{\gamma'}) \in H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, V_{L_\lambda}(\psi)) \oplus H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, \iota(V_{L_\lambda}(\psi))).$$

Let us write pr_1 and pr_2 for the projections to the two direct summands above. By Corollary 2.28, the projection $\mathrm{pr}_1 \, \mathrm{loc}_p(\mathbf{z}^{\mathrm{Kato}}_{\gamma'})$ to $H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, V_{L_\lambda}(\psi))$ is

$$\mathrm{loc}_{\mathfrak{p}}\left(\mathbf{e}_\infty(\gamma) \otimes (\zeta_{p^n})^{\otimes(-1)}\right).$$

By Proposition 2.29, we see that the projection of $\mathrm{loc}_p(\mathbf{z}^{\mathrm{Kato}}_{\gamma'})$ to the other direct summand is

$$\delta \cdot \mathrm{loc}_{\mathfrak{p}}\left[\iota_*\left(\mathbf{e}_\infty(\gamma) \otimes (\zeta_{p^n})^{\otimes(-1)}\right)\right] = \left[\delta \cdot \mathrm{loc}_{\mathfrak{p}}\left(\iota_*(\mathbf{e}_\infty(\gamma))\right)\right] \otimes (\zeta_{p^n})^{\otimes(-1)}.$$

We have

$$\iota_*\left(\mathbf{e}_\infty(\gamma)\right) = \mathbf{e}_\infty(\iota\gamma),$$

so this simplifies to

$$\mathrm{pr}_2\left(\mathrm{loc}_p \, \mathbf{z}^{\mathrm{Kato}}_{\gamma'}\right) = \delta \cdot \left[\mathrm{loc}_{\mathfrak{p}}\left(\mathbf{e}_\infty(\iota\gamma)\right)\right] \otimes (\zeta_{p^n})^{\otimes(-1)}.$$

Definition 3.1. *Let $L^\gamma_{p,1} \in \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} \mathbb{D}_{\mathrm{cris}}(V_{L_\lambda}(\psi)(k-1))$ and $L^\gamma_{p,2} \in \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} \mathbb{D}_{\mathrm{cris}}(\iota V_{L_\lambda}(\psi)(k-1))$ be the unique elements such that*

$$\mathcal{L}^\Gamma_{V_{L_\lambda}(f)(k-1)}\left(\mathbf{z}^{\mathrm{Kato}}_{\gamma'} \otimes (\zeta_{p^n})^{\otimes(k-1)}\right) = L^\gamma_{p,1} \oplus L^\gamma_{p,2}.$$

We shall see below that if $g = \bar{f}$ is the complex conjugate of f, then $L^\gamma_{p,1}$ will be the ordinary p-adic L-function of g, and $L^\gamma_{p,2}$ is the critical-slope p-adic L-function of g.

Theorem 3.2. *For every character η of Γ, we have*

$$L^\gamma_{p,1}(\eta) = \mathbb{L}_{\mathfrak{f}p^\infty}\left(\eta\left(\psi_\lambda \chi^{k-2}\right)^{-1}\right) \cdot t^{k-1}\gamma,$$

and

$$L^\gamma_{p,2}(\eta) = \left(\ell_0 \ldots \ell_{k-2}\delta\mathbb{L}_{\mathfrak{f}p^\infty}\right)\left(\eta\left(\psi^\iota_\lambda \chi^{k-2}\right)^{-1}\right) \cdot \iota\gamma.$$

Proof. For brevity, we shall write e_j for $(\zeta_{p^n})^{\otimes j}$, considered as a basis vector of $\mathbb{Q}_p(j)$.

It is easy to see that if ξ is a character of $G_{\mathfrak{f}p^\infty}$ of the form $\chi^j\tau$, where τ is unramified and $j \geq 0$, and V is any crystalline representation with non-negative

Hodge-Tate weights, then for any $x \in H^1_{\mathrm{Iw}}(K(\mathfrak{f}p^\infty), V)$ and any choice of basis e_ξ of $\mathbb{Q}_p(\xi)$ we have

$$\mathcal{L}_{V(\xi)}^{G_{\mathfrak{f}p^\infty}}(x \otimes e_\xi)(\eta) = (\ell_0 \dots \ell_{j-1})(\eta) \cdot \mathcal{L}_V^{G_{\mathfrak{f}p^\infty}}(x)(\eta\xi^{-1}) \otimes t^{-j} e_\xi.$$

Note that if ξ takes values in the finite extension $L/\mathbb{Q}_p$, this is an equality of two elements of $L \otimes \widehat{F}_\infty \otimes \mathbb{D}_{\mathrm{cris}}(V(\xi))$: the element $t^{-j} e_\xi \in \mathbb{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} L(\xi)$ transforms via τ under $G_{\mathbb{Q}_p}$, and hence lies in $\widehat{F}_\infty \otimes \mathbb{D}_{\mathrm{cris}}(L(\xi))$, since the periods of unramified characters lie in $\widehat{F}_\infty \subseteq \mathbb{B}_{\mathrm{cris}}$.

We apply this result with $V = \mathbb{Q}_p$ (the trivial representation), $x = \mathbf{e}_{\mathfrak{f}p^\infty} \otimes e_{-1}$, and various values of ξ. Firstly, taking ξ to be the cyclotomic character, we have

$$\mathbb{L}_{\mathfrak{f}p^\infty} = \ell_0^{-1} \mathcal{L}_{\mathbb{Q}_p(1)}^{G_{\mathfrak{f}p^\infty}}(\mathbf{e}_{\mathfrak{f}p^\infty}),$$

and thus

$$\mathbb{L}_{\mathfrak{f}p^\infty}(\eta) = \mathcal{L}_{\mathbb{Q}_p}^{G_{\mathfrak{f}p^\infty}}(\mathbf{e}_{\mathfrak{f}p^\infty} \otimes e_{-1})(\chi^{-1}\eta) \otimes t^{-1} e_1. \tag{4}$$

On the other hand we have

$$\begin{aligned} L_{p,1}^\gamma(\eta) &= \mathcal{L}^{\Gamma}_{V_{L_\lambda}(\psi)(k-1)}\left(\mathrm{pr}_1(\mathbf{z}^{\mathrm{Kato}}_{\gamma'}) \otimes e_{k-1}\right)(\eta) \\ &= \mathcal{L}^{G_{\mathfrak{f}p^\infty}}_{V_{L_\lambda}(\psi)(k-1)}\left(\mathbf{e}_\infty(\gamma) \otimes e_{k-2}\right)(\eta) \end{aligned}$$

The group $G_{\mathbb{Q}_p}$ acts on $V_{L_\lambda}(\psi)(k-1)$ via the unramified character $\chi^{k-1}\psi_{L_\lambda}$, so this is

$$L_{p,1}^\gamma(\eta) = \mathcal{L}_{\mathbb{Q}_p}^{G_{\mathfrak{f}p^\infty}}(\mathbf{e}_\infty \otimes e_{-1})\left((\chi^{k-1}\psi_{L_\lambda})^{-1}\eta\right) \otimes (t^{k-1}\gamma) \otimes (t^{1-k} e_{k-1}).$$

Comparing this with (4), we deduce that

$$L_{p,1}^\gamma(\eta) = \mathbb{L}_{\mathfrak{f}p^\infty}\left((\chi^{k-2}\psi_{L_\lambda})^{-1}\eta\right) \otimes (t^{k-1}\gamma) \otimes (t^{2-k} e_{k-2}).$$

If we identify $\mathbb{D}_{\mathrm{cris}}(\mathbb{Q}_p(k-2))$ with $\mathbb{Q}_p$ in the usual way, $t^{2-k}e_{k-2}$ is sent to 1. As remarked above, the element $t^{k-1}\gamma \in \mathbb{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_{L_\lambda}(\psi)$ lies in $\widehat{F}_\infty \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(V_{L_\lambda}(\psi))$. So if ω is a K-basis of $S(\psi)$, then the image of ω under the crystalline comparison isomorphism is a basis of $\mathbb{D}_{\mathrm{cris}}(V_{L_\lambda}(\psi))$, and if we define $\Omega_p = (\gamma \otimes e_{1-k})/\omega$, this will lie in $\widehat{F}_\infty$ and our result becomes

$$L_{p,1}^\gamma(\eta) = \mathbb{L}_{\mathfrak{f}p^\infty}\left((\chi^{k-2}\psi_{L_\lambda})^{-1}\eta\right) \cdot \Omega_p \omega.$$

We now turn to $L_{p,2}^\gamma$. We have

$$\begin{aligned} L_{p,2}^\gamma(\eta) &= \mathcal{L}^{\Gamma}_{\iota(V_{L_\lambda}(\psi))(k-1)}\left(\mathrm{pr}_2(\mathbf{z}^{\mathrm{Kato}}_{\gamma'}) \otimes e_{k-1}\right)(\eta) \\ &= \mathcal{L}^{G_{\mathfrak{f}p^\infty}}_{\iota(V_{L_\lambda}(\psi))(k-1)}\left((\delta \cdot \mathbf{e}_\infty(\iota\gamma)) \otimes e_{k-2}\right)(\eta) \\ &= (-1)^{k-2}\eta(\delta) \mathcal{L}^{G_{\mathfrak{f}p^\infty}}_{\iota(V_{L_\lambda}(\psi))(k-1)}\left(\mathbf{e}_\infty(\iota\gamma) \otimes e_{k-2}\right)(\eta). \end{aligned}$$

The group $G_{\mathbb{Q}_p}$ acts on $\iota(V_{L_\lambda}(\psi))$ by the character ψ_λ^ι, which is unramified; so this is

$$\begin{aligned} L_{p,2}^\gamma(\eta) &= (-1)^{k-2}\eta(\delta)(\ell_0 \dots \ell_{k-2})(\eta) \cdot \mathcal{L}_{\mathbb{Q}_p}^{G_{\mathfrak{f}p^\infty}}(\mathbf{e}_\infty \otimes e_{-1})\left((\chi^{k-1}\psi_\lambda^\iota)^{-1}\eta\right) \\ &\qquad \otimes t^{1-k} e_{k-1} \otimes \iota\gamma. \\ &= (-1)^{k-2}\eta(\delta)(\ell_0 \dots \ell_{k-2})(\eta) \cdot \mathbb{L}_{\mathfrak{f}p^\infty}\left((\chi^{k-2}\psi_{L_\lambda}^\iota)^{-1}\eta\right) \otimes t^{2-k} e_{k-2} \otimes \iota\gamma. \end{aligned}$$

As above, we identify $t^{2-k}e_{k-2} \in \mathbb{D}_{\mathrm{cris}}(\mathbb{Q}_p(k-2))$ with $1 \in \mathbb{Q}_p$; and if ω is a basis of $S_L(\psi)$, the image of $\iota\omega$ under the comparison isomorphism is a basis of $\mathbb{D}_{\mathrm{cris}}(\iota(V_{L_\lambda}(\psi)))$, so if we define $\Omega_p^\iota = (\iota\gamma)/(\iota\omega)$ this becomes

$$L_{p,2}^\gamma(\eta) = (-1)^{k-2}\eta(\delta)(\ell_0 \dots \ell_{k-2})(\eta) \cdot \mathbb{L}_{\mathfrak{f}p^\infty}\left((\chi^{k-2}\psi_{L_\lambda})^{-1}\eta\right) \cdot \Omega_p^\iota \iota\omega.$$

□

Definition 3.3. *Let ω be a basis of $S_L(\psi)$ as above, and let $L_{p,\alpha}(g)$ and $L_{p,\beta}(g)$ be the elements of $\mathcal{H}_{L_\lambda}(\Gamma)$ defined by*

$$L_{p,1}^{\gamma} = L_{p,\alpha}(g) \cdot \omega$$

and

$$L_{p,2}^{\gamma} = L_{p,\beta}(g) \cdot \iota\omega.$$

Then $L_{p,\alpha}$ and $L_{p,\beta}$ are the p-adic L-functions attached to g, where α and β are respectively the unit and non-unit roots of the Hecke polynomial of g.

As shown in [Kat04, §16], this is consistent with the classical Amice–Velu–Vishik construction of the ordinary p-adic L-function $L_{p,\alpha}(g)$, and thus it is natural to regard $L_{p,\beta}(g)$ as a candidate for a critical-slope p-adic L-function. This is the definition of the Kato critical-slope L-function used in [LZ11a].

Theorem 3.4. *Up to multiplication by two nonzero scalars, one for each sign, $L_{p,\beta}(g)$ coincides with the modular symbol critical-slope L-function $L_{p,\beta}^{\mathrm{MS}}(g)$ attached to the non-ordinary p-stabilization of f in [Bel11b].*

Proof. This follows by comparing the formulae of Theorem 3.2 with Theorem 2 of [Bel11b]. Note that Bellaïche shows that if ρ_1 and ρ_2 are the two characters by which $\mathrm{Gal}(\overline{K}/K)$ acts on V_g^*, then

$$\begin{cases} L_{p,\alpha}(g)(\eta) & = \mathbb{L}_{\mathfrak{f}p^\infty}(\rho_2\eta^{-1}) \cdot (\mathrm{constant}^{\pm}), \\ L_{p,\beta}^{\mathrm{MS}}(g)(\eta) & = (\ell_0 \cdots \ell_{k-2})(\eta) \cdot \mathbb{L}_{\mathfrak{f}p^\infty}(\rho_1\eta^{-1}) \cdot (\mathrm{constant}^{\pm}). \end{cases}$$

Here constant$^{\pm}$ indicates an equality of distributions on Γ up to multiplication by two nonzero constants (one for each sign). On the other hand, we have proved that

$$\begin{cases} L_{p,\alpha}(g)(\eta) & = \mathbb{L}_{\mathfrak{f}p^\infty}(\chi\rho_1^{-1}\eta) \cdot (\mathrm{constant}), \\ L_{p,\beta}(g)(\eta) & = (\ell_0 \cdots \ell_{k-2})(\eta) \cdot \mathbb{L}_{\mathfrak{f}p^\infty}(\chi\rho_2^{-1}\eta) \cdot (\mathrm{constant}). \end{cases}$$

To reconcile these formulae, we note that the p-adic L-function $\mathbb{L}_{\mathfrak{f}p^\infty}$ satisfies a functional equation [dS87, §II.6]

$$\mathbb{L}_{\mathfrak{f}p^\infty}(\iota(\eta)) = C(\eta) \cdot \mathbb{L}_{\mathfrak{f}p^\infty}(\chi\eta^{-1}),$$

for a function $C(\eta)$ (involving a p-adic root number and various other correction terms) which depends only on the coset of η modulo characters factoring through $\mathrm{Gal}(\mathbb{Q}_\infty^+/\mathbb{Q})$. Since $\iota(\rho_1) = \rho_2$ and vice versa, we deduce that

$$L_{p,\beta}(g) = L_{p,\beta}^{\mathrm{MS}}(g) \cdot (\mathrm{constant}^{\pm}).$$

Since the modular symbol L-function is only defined up to scalars, this completes the proof. □

Note 3.5. *Both Kato's and Bellaïche's critical-slope p-adic L-functions are only defined up to multiplication by a nonzero constant for characters of each sign; in Kato's construction these constants correspond to the choice of γ, whose projection to each of the $\pm$ eigenspaces of complex conjugation must be non-zero. It seems natural to ask whether one can choose normalizations for both in a compatible fashion so Theorem 3.4 holds exactly, but the present authors do not feel sufficiently familiar with the modular symbol construction to comment further.*

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