Coleman Maps for Modular Forms at Supersingular Primes over Lubin-Tate Extensions

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Abstract

Given an elliptic curve with supersingular reduction at an odd prime p, Iovita and Pollack have generalised results of Kobayashi to define even and odd Coleman maps at p over Lubin-Tate extensions given by a formal group of height 1. We generalise this construction to modular forms of higher weights.

0 Introduction

Let f be a normalised eigen-newform of integral weight at least 2 and p an odd supersingular prime for f (i.e. p divides a_p but not the level of f). On the one hand, the p-adic L-functions of f defined in [11] have unbounded coefficients. On the other hand, the p-Selmer group over the $\mathbb{Q}_\infty$, the extension of $\mathbb{Q}$ by adjoining all p power roots of unity, is not Λ-cotorsion where Λ is the Iwasawa algebra of $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$, which can be identified with the set of power series over $\mathbb{Z}_p[\mathrm{Gal}(k_1/\mathbb{Q}_p)]$. It makes the Iwasawa theory for f at p difficult.

Much progress has been made in this direction. In [13], Pollack has defined the plus and minus analytic p-adic L-functions $L_p^{\pm}$, which have bounded coefficients in the case $a_p = 0$. When f corresponds to an elliptic curve E defined over $\mathbb{Q}$ and p is as above, Kobayashi [8] defined the even and odd Selmer groups $\mathrm{Sel}_p^{\pm}(E/\mathbb{Q}_\infty)$ by modifying the local condition of the usual Selmer group at p.

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These conditions are obtained by applying Honday theory to the formal group associated to E at p. Kobayashi then used these conditions to construct

$$\mathrm{Col}^{\pm} : \lim_{\leftarrow} H^1(k_n, T_E) \to \Lambda$$

where T_E is the Tate module of E at p and $k_n = \mathbb{Q}_p(\mu_{p^n})$. It turns out that on applying $\mathrm{Col}^{\pm}$ to the Kato zeta element defined in [6], one obtains $L_p^{\pm}$, which can be used to show that $\mathrm{Sel}_p^{\pm}(E/\mathbb{Q}_\infty)$ are Λ-cotorsion. It is then possible to formulate the "main conjecture" in the following form:

Conjecture 0.1. *With the notation above, the characteristic ideal of the Pontryagin dual of* $\mathrm{Sel}_p^{\pm}(E/\mathbb{Q}_\infty)$ *is generated by* $L_p^{\pm}$.

On the one hand, the construction of $\mathrm{Col}^{\pm}$ was generalised by Iovita and Pollack [5] to Lubin-Tate extensions given by formal groups of height 1. That is, we can replace k_n by extensions of $\mathbb{Q}_p$ obtained by adjoining torsion points of a Lubin-Tate group of height 1 defined over $\mathbb{Z}_p$. On the other hand, Kobayashi's construction can be generalised to modular forms of higher weights by using Perrin-Riou's exponential map (see [10]). We will show that one can generalise the construction of the former to higher weight modular forms as well by using the Perrin-Riou's exponential map constructed by Zhang [15].

As in [10], instead of defining the Coleman maps using local conditions obtained from the formal group, we define the Coleman maps directly using the Perrin-Riou's exponential. We then obtain our new local conditions from $\ker(\mathrm{Col}^{\pm})$, which turn out to agree with the ones given by Kobayashi and Iovita-Pollack. We then use these conditions to define the corresponding Selmer groups.

We now outline the construction of $\mathrm{Col}^{\pm}$ here. Let V_f be the Deligne p-adic representation of $G_{\mathbb{Q}}$ associated to f. Write $V = V_f(1)$, the Tate twist of V_f and fix T a lattice in V which is stable under $G_{\mathbb{Q}}$. Then, the Perrin-Riou's exponential map enables us to define two elements

$$\mathbb{E}_{h,V}(\mu_{\xi^{\pm}}) \in \mathcal{H}_{(k-1)/2} \otimes \lim_{\leftarrow} H^1(k_n, T)$$

where $\mathcal{H}_{(k-1)/2}$ denotes the set of power series over $\mathbb{Q}_p[\mathrm{Gal}(k_1/\mathbb{Q}_p)]$ which are of order $\log_p^{(k-1)/2}$. We then define

$$\begin{aligned} \mathcal{L}_{\xi^{\pm}} : \lim_{\leftarrow} H^1(k_n, T^*(1)) &\to \mathcal{H}_{(k-1)/2} \\ \mathbf{z} &\mapsto < \mathbb{E}_{h,V}(\mu_{\xi^{\pm}}), \mathbf{z} > \end{aligned}$$

where $<,>$ is a pairing on

$$\Big(\mathcal{H}_{(k-1)/2} \underset{\Lambda}{\otimes} \lim_{\leftarrow} H^1(k_n, T)\Big) \times \lim_{\leftarrow} H^1(k_n, T^*(1)) \to \mathcal{H}_{(k-1)/2}.$$

On computing some of its special values, we show that $\mathcal{L}_{\xi^{\pm}}(\mathbf{z})$ is divisible by $\log_{p,k}^{\pm}$, which is defined in [13] and has exact order $\log_p^{(k-1)/2}$. This enables us to define

$$\begin{aligned} \mathrm{Col}^{\pm} : \varprojlim H^1(k_n, T^*(1)) &\rightarrow \mathbb{Q} \otimes \Lambda \\ \mathbf{z} &\mapsto \mathcal{L}_{\xi^{\pm}}(\mathbf{z}) / \log^{\pm}_{p,k} . \end{aligned}$$

The structure of this paper is as follows. We will review results of [15] in Section 1. In particular, we will state the properties of the Perrin-Riou's exponential map which we will need for our construction of the Coleman maps. In Section 2, we will construct the Coleman maps using ideas from [10]. The kernels and images of these maps will be described in Section 3 under certain technical assumptions. In particular, we will define the even and odd Selmer groups for some $\mathbb{Z}_p$-extensions of a number field using our description of the kernels. Finally, we explain how the construction in Section 2 can be generalised to relative Lubin-Tate groups in Section 4 using ideas of Kim (see [7]).

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1 Perrin-Riou's exponential map over height 1 Lubin-Tate extensions

In [15], Zhang has generalised the construction of Perrin-Riou's exponential map defined in [12] to Lubin-Tate extensions. We review his results here.

We fix an odd prime p and π a uniformiser of $\mathbb{Z}_p$. Let α be the p-adic unit in $\mathbb{Z}_p^\times$ such that $\pi = \alpha p$. Let g be a lift of Frobenius with respect to π, i.e. a power series over $\mathbb{Z}_p$ such that $g(X) = \pi X +$(higher terms) and $g(X) \equiv X^p \mod p$. Then, g gives rise to an one-dimensional height-one formal group over $\mathbb{Z}_p$, which is independent of the choice of g up to isomorphism over $\mathbb{Z}_p$. We denote this formal group by $\mathcal{F}$.

We write $K = \mathbb{Q}_p$ (reason being we want to replace $\mathbb{Q}_p$ by a finite unramified extension of $\mathbb{Q}_p$ in Section 4), K_n denotes the extension of K obtained by adjoining the π^nth roots of $\mathcal{F}$ and G_n denotes the Galois group of K_n over K for $0 \leq n \leq \infty$. In particular, $G_n \cong (\mathbb{Z}/p^n)^\times$ and $G_\infty \cong G_1 \times \mathrm{Gal}(K_\infty/K_1) \cong \mathbb{Z}/(p-1) \times \mathbb{Z}_p$.

Let κ be the character of G_K (the absolute Galois group of K) given by its action on the Tate module of $\mathcal{F}$. Then, $\sigma\omega = [\kappa(\sigma)]_{\mathcal{F}}(\omega)$ for all $\omega \in \mathcal{F}[\pi^\infty]$. If χ denotes the cyclotomic character of G_K, then $\kappa = \chi\psi$ for an unramified character ψ.

Let Ξ denote the completion of the maximal unramified extension of $\mathbb{Q}_p$ and $\mathfrak{O}$ its ring of integers. Let $\eta : \mathbb{G}_m \rightarrow \mathcal{F}$ be an isomorphism between the multiplicative group and $\mathcal{F}$. Then $\eta \in \mathfrak{O}[[X]]$. Moreover, $\eta(X) = \Omega X+$ (higher degree terms), where Ω is a p-adic unit. The lift of Frobenius g satisfies $g \circ \eta = \eta^\varphi \circ ((1+X)^p - 1)$ where φ is the Frobenius of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ which acts on η by acting on its coefficients. In particular, $\Omega^\varphi = \alpha\Omega$.

Definition 1.1. *We define $\Xi[[X]]^\psi$ to be the set of power series f, defined over Ξ, such that $\sigma f(X) = f((1+X)^{\psi(\sigma)} - 1) \forall \sigma \in G_K$.*

In particular, [15, (1.13)] says that $\eta \in \Xi[[X]]^\psi$. The significance of this set is given by the following:

Lemma 1.2. *Let $f \in \Xi[[X]]^\psi$ and ζ a p^nth root of unity. Then $f(\zeta - 1) \in K_n$.*

Proof. By definition, $\sigma f(X) = f((1+X)^{\psi(\sigma)} - 1)$ for any $\sigma \in G_K$. Therefore, we have

$$\begin{aligned} \sigma(f(\zeta - 1)) &= (\sigma f)(\zeta^\sigma - 1) \\ &= f(\zeta^{\chi(\sigma)\psi(\sigma)} - 1) \\ &= f(\zeta^{\kappa(\sigma)} - 1). \end{aligned}$$

If, in addition, $\sigma \in G_{K_n}$, then $\kappa(\sigma) \in 1 + p^n\mathbb{Z}_p$. Hence, $\sigma(f(\zeta-1)) = f(\zeta - 1)$ for any $\sigma \in G_{K_n}$, so we are done. □

From now on, we fix a primitive p^nth root of unity ζ_{p^n} for each positive integer n such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. This determines an element $t \in B_{dR}^+$ (see [2, Section III.1] for details). We also fix a crystalline (hence de Rham) representation V of G_K and write $D(V) = D_{dR}(V) = D_{\mathrm{cris}}(V)$ for its Dieudonné module which is equipped with a de Rham filtration and an action of φ. We denote the ith de Rham filtration by $D^i(V)$. We write $r(V)$ for the slope of φ on $D(V)$. Note that the action of φ extends to $\Xi \otimes D(V)$ naturally.

We write $V(k)$ for the kth Tate twist of V. Then, $D(V(k)) = t^{-k}D(V)$ as G_K acts on t via χ. Similarly, $D(V(\kappa^k)) = t_\pi^{-k}D(V)$ where $t_\pi = \Omega t$ since G_K acts on t_π via κ by [15, Section 2]. Their filtrations are given by the following:

Lemma 1.3. *The de Rham filtrations satisfy*

$$D^i(V(\kappa^j)) = D^i(V(j)) = t_\pi^{-j}D^{i+j}(V).$$

Proof. Since $\Omega \in \bar{K}^\times$, we have

$$\begin{aligned} D^i(V(\kappa^j)) &= (t_\pi^{-j}D(V)) \cap t^i B_{dR}^+ \\ &= t_\pi^{-j}(D(V) \cap t^{i+j}\Omega^j B_{dR}^+) \\ &= t_\pi^{-j}(D(V) \cap t^{i+j} B_{dR}^+) \\ &= t_\pi^{-j}D^{i+j}(V). \end{aligned}$$

Hence we are done. □

For $r \in \mathbb{R}_{\geq 0}$, let B be a Banach p-adic space, then $\mathcal{D}_r(\mathbb{Q}_p, B)$ denotes the set of tempered B-valued distributions of order r (in the sense of [2, Definition I.4.2]) on the locally analytic functions with compact support in $\mathbb{Q}_p$. It is equipped with a Galois action of G_K as defined in [15, (3.1)]. Similarly, if A is a compact open subset of $\mathbb{Q}_p$, $\mathcal{D}_r(A, B)$ denotes the set of tempered distributions of order r on A with values in B.

When $A=\mathbb{Z}_p$, we write the Amice transform of $\mu \in \mathcal{D}_r(\mathbb{Z}_p, B)$ as $\mathcal{A}_\mu \in B[[X]]$, i.e.

$$\mathcal{A}_\mu(X) = \int_{\mathbb{Z}_p} (1+X)^x \mu(x).$$

We define $\mathcal{D}_r(\mathbb{Q}_p, \Xi \otimes D(V))^\psi$ to be the subset of $\mathcal{D}_r(\mathbb{Q}_p, \Xi \otimes D(V))$ consisting of all the distributions μ satisfying:

$$\sigma\left(\int_{\mathbb{Q}_p} f\mu\right) = \int_{\mathbb{Q}_p} f(\psi(\sigma)x)\mu \ \forall \sigma \in G_K.$$

Remark 1.4. *Let $\mu \in \mathcal{D}_r(\mathbb{Z}_p, \Xi \otimes D(V))$. Then, $\mu \in \mathcal{D}_r(\mathbb{Z}_p, \Xi \otimes D(V))^\psi$ iff its Amice transform is in $\Xi[[X]]^\psi \otimes D(V)$ (see [15, Proposition 2.4(i)]).*

We define $\widetilde{\mathcal{D}_r}(\mathbb{Z}_p^\times, \Xi \otimes \mathcal{D}(V))$ to be $\varprojlim_{\mathrm{Tw}} \mathcal{D}_r\left(\mathbb{Z}_p^\times, \Xi \otimes D(V(\kappa^k))\right)$ where Tw is the twist map given by $\mu \mapsto (-tx)^{-1}\mu$, which is well defined by [14, Lemma 3.6]. We define $\widetilde{\mathcal{D}_r}(\mathbb{Q}_p, \Xi \otimes \mathcal{D}(V))$ similarly. In [15, Theorems 3.3 and 3.6], the generalised Perrin-Riou's exponential is given by:

Theorem 1.5. *Let h be a positive integer such that $D^{-h}(V) = D(V)$. Then, there is a map*

$$\mathbb{E}_{h,V} : \widetilde{\mathcal{D}_r}(\mathbb{Q}_p, \Xi \otimes D(V))^{\varphi_{\mathcal{D}} \otimes \varphi = 1, \psi} \to H^1\left(K_\infty, \mathcal{D}_{r+r(V)+h}(\mathbb{Z}_p^\times, D(V))\right)^{G_\infty}$$

such that for $k \geq 1-h$

$$\begin{aligned}
\int_{\mathbb{Z}_p^\times} x^k \mathbb{E}_{h,V}(\mu) &= (k+h-1)! \exp_k\left((1-\varphi)^{-1}(1-\frac{\varphi^{-1}}{p})\int_{\mathbb{Z}_p^\times} \frac{\mu}{(-tx)^k}\right), \\
\int_{1+p^n\mathbb{Z}_p} x^k \mathbb{E}_{h,V}(\mu) &= (k+h-1)! \exp_k\left(\frac{\varphi^{-n}}{p^n}\int_{\mathbb{Z}_p} \epsilon\left(\frac{x}{p^n}\right)\frac{\mu}{(-tx)^k}\right)
\end{aligned}$$

where ϵ is as defined in [2, Section V.1] and $\exp_k$ denotes the exponential map for the p-adic representation $V(\kappa^k)$ as defined in [1].

2 The construction of even and odd Coleman maps

We construct $\mathrm{Col}^\pm$ in three steps. First, we prove some elementary results about distributions on $\mathbb{Z}_p^\times$ in Section 2.1. In Section 2.2, we explain how to construct a measure $\mu_\xi \in \mathcal{D}_0(\mathbb{Z}_p^\times, \Xi \otimes D(V))^\psi$ from a given $\xi \in D(V)$ and compute some special values of $\mathbb{E}_{h,V}(\mu_\xi)$ using Theorem 1.5 and results from Section 2.1 . Finally, in Section 2.3, we apply these results to a modular form f by choosing two elements of $D(V_f)$, namely $\xi^\pm$. We then proceed as explained in the introduction to construct $\mathrm{Col}^\pm$.

2.1 Distributions on $\mathbb{Z}_p^\times$

Let $\mu \in \mathcal{D}_r(\mathbb{Z}_p, \Xi \otimes \mathcal{D}(V))^\psi$, then $\mu \in \mathcal{D}_r(\mathbb{Z}_p^\times, \Xi \otimes \mathcal{D}(V))^\psi$ iff

$$\sum_{\zeta^p=1} \mathcal{A}_\mu(\zeta(1+X)-1) = 0.$$

On the space of power series satisfying this condition, $D = (1+X)\frac{d}{dX}$ acts bijectively. Moreover, for such μ, we have

$$D^k \mathcal{A}_\mu(\zeta_{p^n} - 1) = \int_{\mathbb{Z}_p^\times} \epsilon\left(\frac{x}{p^n}\right) x^k \mu, \tag{1}$$

see e.g. [2, Section I.5].

Lemma 2.1. *Any $\mu \in \mathcal{D}_r(\mathbb{Z}_p^\times, \Xi \otimes \mathcal{D}(V))^\psi$ can be lifted to*

$$\widetilde{\mu} \in \widetilde{\mathcal{D}_r}(\mathbb{Q}_p, \Xi \otimes \mathcal{D}(V))^{\varphi_{\mathcal{D}} \otimes \varphi = 1, \psi}.$$

Moreover, the image of such a lift under $\mathbb{E}_{h,V}$ is independent of the choice of the lift.

Proof. [2, Lemma IX.2.8 and Remark IX.2.6(iii)] and [15, Lemma 3.5]. □

Given any $\mu \in \mathcal{D}_r(\mathbb{Z}_p^\times, \Xi \otimes \mathcal{D}(V))^\psi$, we abuse notation and write $\mathbb{E}_{h,V}(\mu) = \mathbb{E}_{h,V}(\widetilde{\mu})$ where $\widetilde{\mu}$ is a lift given by Lemma 2.1. The fact that $\varphi_{\mathcal{D}} \otimes \varphi(\widetilde{\mu}) = \widetilde{\mu}$ implies that

$$\int_{pA} f(x)\widetilde{\mu} = \varphi\left(\int_A f(px)\widetilde{\mu}\right) \tag{2}$$

for any f and $A \subset \mathbb{Q}_p$. It allows us to compute some special values of $\widetilde{\mu}$.

Lemma 2.2. *With the above notation, $\int_{\mathbb{Z}_p} x^k \widetilde{\mu} = (1-p^k\varphi)^{-1}\left(D^k\mathcal{A}_\mu(0)\right)$.*

Proof. Since $\widetilde{\mu_\xi}$ restricts to μ_ξ on $\mathbb{Z}_p^\times$, (1) implies that

$$\int_{\mathbb{Z}_p^\times} x^k \widetilde{\mu_\xi} = \int_{\mathbb{Z}_p^\times} x^k \mu_\xi = D^k \mathcal{A}_\mu(0).$$

Hence, by applying (2) to the decomposition

$$\int_{\mathbb{Z}_p} x^k \widetilde{\mu} = \int_{p\mathbb{Z}_p} x^k \widetilde{\mu} + \int_{\mathbb{Z}_p^\times} x^k \widetilde{\mu},$$

we have

$$\int_{\mathbb{Z}_p} x^k \widetilde{\mu} = p^k \varphi\left(\int_{\mathbb{Z}_p} x^k \widetilde{\mu}\right) + D^k \mathcal{A}_\mu(0).$$

□

Lemma 2.3. *With the notation above,*

$$\int_{\mathbb{Z}_p} \epsilon\left(\frac{x}{p^n}\right) x^k \widetilde{\mu} = \sum_{i=0}^{n-1} p^{ik}\varphi^i \left(D^k \mathcal{A}_\mu(\zeta_{p^{n-i}} - 1)\right) + p^{nk}(1-p^k\varphi)^{-1}(D^k\mathcal{A}_\mu(0)).$$

Proof. Since $\mathbb{Z}_p = \mathbb{Z}_p^\times \cup p\mathbb{Z}_p^\times \cup \cdots \cup p^{n-1}\mathbb{Z}_p^\times \cup p^n\mathbb{Z}_p$, we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \epsilon\left(\frac{x}{p^n}\right) x^k \widetilde{\mu} \\ = & \sum_{i=0}^{n-1} \int_{p^i\mathbb{Z}_p^\times} \epsilon\left(\frac{x}{p^n}\right) x^k \mu + \int_{p^n\mathbb{Z}_p} \epsilon\left(\frac{x}{p^n}\right) x^k \widetilde{\mu} \\ = & \sum_{i=0}^{n-1} p^{ik}\varphi^i \left(\int_{\mathbb{Z}_p^\times} \epsilon\left(\frac{x}{p^{n-i}}\right) x^k \mu\right) + p^{nk}\varphi^n \int_{\mathbb{Z}_p} x^k \widetilde{\mu} \end{aligned}$$

where the last equality follows from repeated applications of (2). Hence the result by (1) and Lemma 2.2. □

2.2 Computing some special values

With the notation above, we define

$$\bar{\eta}(X) = \eta(X) - \frac{1}{p}\sum_{\zeta^p=1} \eta(\zeta(1+X)-1).$$

Then $\sum_{\zeta^p=1} \bar{\eta}(\zeta(1+X)-1) = 0$. Moreover, we have:

Lemma 2.4. *We have $\bar{\eta} \in \Xi[[X]]^\psi$.*

Proof. Let $\sigma \in G_{\mathbb{Q}_p}$ and ζ a pth root of unity. By [15, (1.13)], $\eta \in \Xi[[X]]^\psi$ and $\sigma\eta(X) = \eta((1+X)^{\psi(\sigma)} - 1)$. If we replace X by $\zeta^\sigma(1+X) - 1$, we have

$$\begin{aligned} \sigma(\eta(\zeta(1+X)-1)) &= (\sigma\eta)(\zeta^\sigma(1+X)-1) \\ &= \eta((\zeta^\sigma(1+X))^{\psi(\sigma)} - 1) \\ &= \eta(\zeta^{\kappa(\sigma)}(1+X)^{\psi(\sigma)} - 1) \end{aligned}$$

Hence, on summing over $\zeta^p = 1$, we have

$$\begin{aligned} \sigma\left(\sum_{\zeta^p=1} \eta(\zeta(1+X)-1)\right) &= \sum_{\zeta^p=1} \sigma(\eta(\zeta(1+X)-1)) \\ &= \sum_{\zeta^p=1} \eta(\zeta^{\kappa(\sigma)}(1+X)^{\psi(\sigma)} - 1) \\ &= \sum_{\zeta^p=1} \eta(\zeta(1+X)^{\psi(\sigma)} - 1) \text{ (as } \kappa(\sigma) \in \mathbb{Z}_p^\times). \end{aligned}$$

Hence, the sum $\sum_{\zeta^p=1} \eta(\zeta(1+X)-1) \in \Xi[[X]]^\psi$, so we are done. □

Let $\xi \in \mathcal{D}(V)$, then $\bar{\eta}(X) \otimes \xi$ defines an element $\mu_\xi \in \mathcal{D}_0(\mathbb{Z}_p^\times, \Xi \otimes \mathcal{D}(V))$ with

$$\bar{\eta}(X) \otimes \xi = \int_{\mathbb{Z}_p^\times} (1+X)^x \mu_\xi.$$

By Lemma 2.4 and Remark 1.4, $\mu_\xi \in \mathcal{D}_0(\mathbb{Z}_p^\times, \Xi \otimes \mathcal{D}(V))^\psi$. On applying the Perrin-Riou's exponential, we have:

Proposition 2.5. *With the notation above, we have for $n \geq 1$ and $k \geq 1-h$*

$$\int_{1+p^n\mathbb{Z}_p} (-x)^k \mathbb{E}_{h,V}(\mu_\xi) = (k+h-1)! \exp_k(\gamma_{n,k}(\xi))$$

where $\gamma_{n,k}(\xi)$ is defined by

$$\frac{1}{p^n}\left(\sum_{i=0}^{n-i} D^{-k}\bar{\eta}^{\varphi^{i-n}}(\zeta_{p^{n-i}}-1) \otimes \varphi^{i-n}(\xi_k) + (1-\varphi)^{-1}(D^{-k}\bar{\eta}(0) \otimes \xi_k)\right)$$

with $\xi_k = \xi t^{-k}$.

Proof. The result follows from combining Theorem 1.5 with Lemmas 2.2 and 2.3 and the fact that $\varphi(t) = pt$. □

Our assumption on the eigenvalues of φ implies that there is an isomorphism

$$\begin{aligned} H^1(K_\infty, \mathcal{D}_r(Z_p^\times, V))^{G_\infty} &\cong \mathcal{D}_r(G_\infty) \otimes \mathbb{H}^1_{\mathrm{Iw}}(V) \\ \mu &\mapsto \left(\int_{1+p^n\mathbb{Z}_p} \mu\right)_n \end{aligned}$$

where $\mathbb{H}^1_{\mathrm{Iw}}(V) := \varprojlim_{\mathrm{cor}} H^1(K_n, V)$ and $\mathcal{D}_r(G_\infty) = \mathcal{D}_r(G_\infty, \mathbb{Q}_p)$ (see e.g. [2, Proposition 2]). Under this identification, we have

$$\mathbb{E}_{h,V}(\mu_\xi) \in \mathcal{D}_{h+r(V)}(G_\infty) \otimes \mathbb{H}^1_{\mathrm{Iw}}(V).$$

Write $\mathrm{Tw}_k : \mathbb{H}^1_{\mathrm{Iw}}(V) \to \mathbb{H}^1_{\mathrm{Iw}}(V(\kappa^k))$ for the twist map. Recall that $\mathrm{Tw}_k(\mu) = (-tx)^{-k}\mu$, so Proposition 2.5 implies that if $n \geq 1$ and $k \geq 1-h$, the nth component of $\mathrm{Tw}_k(\mathbb{E}_{h,V}(\mu))$ is given by

$$(k+h-1)! \exp_k(\gamma_{n,k}(\xi)) \tag{3}$$

where $\exp_k$ now denotes the exponential map $K_n \otimes \mathcal{D}(V(\kappa^k)) \to H^1(K_n, V(\kappa^k))$.

Recall that $G_\infty \cong G_1 \times \Gamma$ where $\Gamma \cong \mathbb{Z}_p$. We fix a topological generator γ of Γ, then $\mathcal{D}_r(G_\infty)$ can be identified with the set of power series in $\gamma - 1$ over $\mathbb{Q}_p[G_1]$ which are $O(\log_p^r)$.

We now assume that V has a F-vector space structure where F is a finite extension of $\mathbb{Q}_p$ and the action of G_K commutes with the multiplication by F.

Denote the ring of integers of F by $\mathcal{O}_F$. Let $\Lambda = \mathcal{O}_F[[G_\infty]] = \varprojlim \mathcal{O}_F[G_n]$, then there is a pairing

$$\begin{array}{rcl} <,>: \mathbb{H}^1_{\mathrm{Iw}}(V) \times \mathbb{H}^1_{\mathrm{Iw}}(V^*(1)) & \to & \mathbb{Q} \otimes \Lambda \\ ((x_n)_n, (y_n)_n) & \mapsto & \left(\sum_{\sigma \in G_n} [x_n^\sigma, y_n]_n \sigma \right)_n \end{array}$$

where $[,]_n$ is the pairing on $H^1(K_n, V) \times H^1(K_n, V^*(1)) \to F$. It extends to

$$\left(\mathcal{D}_m(G_\infty) \underset{\Lambda}{\otimes} \mathbb{H}^1_{\mathrm{Iw}}(V)\right) \times \left(\mathcal{D}_n(G_\infty) \underset{\Lambda}{\otimes} \mathbb{H}^1_{\mathrm{Iw}}(V^*(1))\right) \to \mathcal{D}_{m+n}(G_\infty)$$

for all $m, n \in \mathbb{R}_{\geq 0}$. This enables us to define the following:

Definition 2.6. *For a fixed $\xi \in D(V)$, we define a map*

$$\begin{array}{rcl} \mathcal{L}^h_\xi : \mathbb{H}^1_{\mathrm{Iw}}(V^*(1)) & \to & \mathcal{D}_{r(V)+h}(G_\infty) \\ \mathbf{z} & \mapsto & < \mathbb{E}_{h,V}(\mu_\xi), \mathbf{z} > . \end{array}$$

Following the calculations of [9], we find that for $n \geq 1$, the nth component of $\mathrm{Tw}_k \mathcal{L}_\xi(\mathbf{z})$ is given by:

$$\begin{array}{rcl} \left(\mathrm{Tw}_k \mathcal{L}^h_\xi(\mathbf{z})\right)_n & = & (h+k-1)! \sum_{\sigma \in G_n} [\exp_k(\gamma_{n,k}(\xi)^\sigma), z_{-k,n}]_n \sigma \\ & = & (h+k-1)! [\sum_{\sigma \in G_n} \gamma_{n,k}(\xi)^\sigma \sigma, \sum_{\sigma \in G_n} \exp^*_k(z^\sigma_{-k,n}) \sigma^{-1}]_n \end{array}$$

where $z_{-k,n}$ denotes the image of $\mathbf{z}$ under

$$\mathbb{H}^1_{\mathrm{Iw}}(V^*(1)) \to \mathbb{H}^1_{\mathrm{Iw}}(V^*(1)(\kappa^{-k})) \to H^1(K_n, V^*(1)(\kappa^{-k}))$$

and Tw_k acts on $\mathcal{D}_{r(V)+h}(G_\infty)$ by $\sigma \mapsto \kappa(\sigma)^k \sigma$ for $\sigma \in G_\infty$.

Let θ be a character on G_n which does not factor through G_{n-1}. Since $D^{-k} \overline{\eta}^{\varphi^{i-n}}(\zeta_{p^{n-i}} - 1) \in K_{n-i}$ by Lemma 1.2, we have

$$\theta \left(\sum_{\sigma \in G_n} \gamma_{n,k}(\xi)^\sigma \sigma \right) = \frac{1}{p^n} \sum_{\sigma \in G_n} D^{-k} \overline{\eta}^{\varphi^{-n}} (\zeta_{p^n} - 1)^\sigma \theta(\sigma) \otimes \varphi^{-n}(\xi_k).$$

Hence, as in [10, Lemma 1.4], we have

$$\begin{aligned} & \frac{1}{(h+k-1)!} \kappa^k \theta(\mathcal{L}^h_\xi(\mathbf{z})) \\ = & \frac{1}{p^n} \left[\sum_{\sigma \in G_n} D^{-k} \overline{\eta}^{\varphi^{-n}} (\zeta_{p^n} - 1)^\sigma \theta(\sigma) \otimes \varphi^{-n}(\xi_k), \sum_{\sigma \in G_n} \exp^*_k(z^\sigma_{-k,n}) \theta(\sigma^{-1}) \right]_n . \end{aligned} \tag{4}$$

2.3 Modular forms

From now on, we fix a normalised newform $f = \sum a_n q^n$ of integral weight $k \geq 2$ with p a supersingular prime for f and $a_p = 0$ (i.e. p divides a_p but not the level of f). We allow the character of f to be arbitrary, but for the sole purpose of easing notation, we assume that the character of f takes value 1 at p. Let V_f be the Deligne representation of $G_{\mathbb{Q}}$ defined in [3]. Let $L = \mathbb{Q}(a_n : n \geq 1)$ be the field of coefficients of f and fix a place of L above p. Then, V is a two-dimensional vector space over $F = L_v$ and the action of $G_{\mathbb{Q}}$ commutes with F. If we take V to be $V_f(1)$, the Frobenius φ on $D(V)$ satisfies

$$\varphi^2 - \frac{a_p}{p}\varphi + p^{k-3} = 0.$$

In particular, $r(V) = (k-1)/2 - 1$ and the assumption that the eigenvalues of φ on $D(V_f)$ are not integral powers of p is automatically satisfied. On taking $h = 1$ in Theorem 1.5 and writing $\mathcal{L}_\xi$ for $\mathcal{L}_\xi^h$, we have $\mathrm{Im}(\mathcal{L}_\xi) \subset \mathcal{D}_{(k-1)/2}(G_\infty)$ for any $\xi \in D(V)$.

The de Rham filtration of $D(V_f)$ is given by

$$D^i(V_f) = D^0(V_f(i)) = \begin{cases} D(V_f) & \text{if } i \leq 0 \\ 0 & \text{if } i \geq k \\ F \cdot \omega & \text{if } 1 \leq i \leq k-1. \end{cases}$$

where ω is any non-zero element of $D^1(V_f) = D^0(V)$. We fix one such ω, this corresponds to a choice of periods for f (see [6]). We have $D^0(V(j)) = D^0(V(\kappa^j)) = F \cdot \omega$ for $0 \leq j \leq k-2$.

Let $\gamma = \kappa(u)$, then we can define $\log^{\pm}_{p,k}$ as in [13]:

$$\begin{aligned} \log^+_{p,k} &= \prod_{j=0}^{k-2} \prod_{n=1}^{\infty} \frac{\Phi_{2n}(\gamma^{-j}u)}{p}, \\ \log^-_{p,k} &= \prod_{j=0}^{k-2} \prod_{n=1}^{\infty} \frac{\Phi_{2n-1}(\gamma^{-j}u)}{p}, \end{aligned}$$

where Φ_m denotes the p^mth cyclotomic polynomial. In particular, the zeros of $\log^+_{p,k}$ are given by $\kappa^j\theta$ where $0 \leq j \leq k-2$ and θ is a character of G_n which does not factor through G_{n-1} with n odd, whereas those of $\log^-_{p,k}$ are characters of the same form but with even n. Moreover, $\log^{\pm}_{p,k}$ have exact order $\log_p^{\frac{k-1}{2}}$. We can now give a generalisation of [10, Lemma 2.2]:

Lemma 2.7. *Let $\xi^+ = \varphi(\omega)$ and $\xi^- = \omega$, then $\log^{\pm}_{p,k} | \mathcal{L}_{\xi^\pm}(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{H}^1_{\mathrm{Iw}}(V^*(1))$.*

Proof. We have $\varphi^{2n}(\omega) \in D^0(V(\kappa^r))$ for all integers n and $0 \leq r \leq k-2$. Therefore, by (4), we have

$$\begin{aligned} \kappa^r\theta(\mathcal{L}_{\xi^+}(\mathbf{z})) &= 0 \quad \text{if } n \text{ is odd}, \\ \kappa^r\theta(\mathcal{L}_{\xi^-}(\mathbf{z})) &= 0 \quad \text{if } n \text{ is even} \end{aligned}$$

where θ is a character of G_n which does not factor through G_{n-1}. Hence, the zeros of $\log_{p,k}^{\pm}$ are also zeros of $\mathcal{L}_{\xi^\pm}(\mathbf{z})$, so we are done. $\square$

In particular, since $\mathcal{L}_{\xi^\pm}(\mathbf{z}) \in \mathcal{D}_{(k-1)/2}(G_\infty)$, we have $\mathcal{L}_{\xi^\pm}(\mathbf{z})/\log_{p,k}^{\pm} = O(1)$. Hence, we have:

Definition 2.8. *The even and odd Coleman maps are defined to be*

$$\begin{aligned} \mathrm{Col}^{\pm} : \mathbb{H}^1_{\mathrm{Iw}}(V^*(1)) &\to \mathbb{Q} \otimes \Lambda \\ \mathbf{z} &\mapsto \mathcal{L}_{\xi^\pm}(\mathbf{z})/\log_{p,k}^{\pm}. \end{aligned}$$

3 Kernel

In this section, we describe the kernels of $\mathrm{Col}^{\pm}$, generalising those given in [5] and use them to define the even and odd Selmer groups. We first give some elementary linear algebra results.

3.1 Linear algebra

For any positive integer n, we write $\pi_n = \eta^{\varphi^{-n}}(\zeta_{p^n} - 1)$. Then, $g^{(n)}(\pi_n) = 0$ where $g^{(n)} = \underbrace{g \circ \cdots \circ g}_{n}$. Moreover, $g(\pi_n) = \pi_{n-1}$ and $K_n = K(\pi_n)$. We will from now on assume g to be a good lift of Frobenius in the sense of [5, Section 4.1]. In particular, we will have to assume $\pi \in p(1 + p\mathbb{Z}_p)$ which would exclude many Lubin-Tate extensions of $\mathbb{Q}_p$. However, if we start with a totally ramified $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$, then we can always assume that it is obtained from such Lubin-Tate extensions (see [5] for details). For $n > 1$, let $\pi'_n = \pi_n - \frac{1}{p}\mathrm{Tr}_{n/n-1}(\pi_n) = \pi_n + 1$ and $\pi'_1 = \pi_1 - \frac{1}{p-1}\mathrm{Tr}_{1/0}(\pi_1) = \pi_1 + \frac{p}{p-1}$. Then, $\mathrm{Tr}_{n/n-1}(\pi'_n) = 0$ for all $n \geq 1$.

Lemma 3.1. *Let $K^{(n)}$ be the kernel of the trace map from K_n to K_{n-1}, then $\{\pi_n'^{\sigma} : \sigma \in G_n\}$ generates $K^{(n)}$ over K.*

Proof. Let $x \in K^{(n)}$. By [5, Proposition 4.4], we have $x \in K[G_n]\pi_n + K_{n-1}$. Since $\mathrm{Tr}_{n/n-1}\pi_n \in K_{n-1}$, we can write $x = \sum_{\sigma \in G_n} a_\sigma \pi_n'^{\sigma} + y$ for some $a_\sigma \in K$ and $y \in K_{n-1}$. Since $\mathrm{Tr}_{n/n-1} x = \mathrm{Tr}_{n/n-1}\pi_n'^{\sigma} = 0$ for all σ, we have $y = 0$. Hence we are done. $\square$

Corollary 3.2. *Let $n \geq 0$ be an integer and $\alpha = \sum_{i=0}^{n} x_i \pi'_i$ for some $x_i \in K$ with $\pi'_0 = 1$. Then, the k-vector space generated by $\{\alpha^\sigma : \sigma \in G_n\}$ is given by $\bigoplus_{i \in S} K^{(i)}$ where $S = \{i : x_i \neq 0\}$ and $K^{(0)} = K$.*

Proof. We proceed by induction on $|S|$. The case $|S| = 1$ follows directly from Lemma 3.1.

Without loss of generality, we assume that $x_n \neq 0$. Let $\beta = \sum_{i=0}^{n-1} x_i \pi'_i$. Then, by induction, $\{\beta^\tau : \tau \in G_{n-1}\}$, generates $\bigoplus_{i \in S \setminus \{n\}} K^{(i)}$ over K. Fix $\tau \in G_{n-1}$ and consider the following p elements: α^σ, $\sigma|_{K_{n-1}} = \tau$. Then, their sum equals $p\beta^\tau + (\mathrm{Tr}_{n/n-1}\, \pi'_n)^\tau = p\beta^\tau$. Therefore, for any $\tau \in G_{n-1}$ and $\sigma \in G_n$, β^τ and π'^σ_n lie inside the K-vector space generated by α^σ. Hence we are done. □

3.2 Description of the kernels

We now fix a lattice T_f in V_f which is stable under G_K. Write $T = T_f(1) \subset V = V_f(1)$. To describe the kernel of $\mathrm{Col}^\pm$, we will assume $p \geq k - 1$ as in [10]. This implies that $(V/T(\kappa^m))^{G_{K_n}} = 0$ for any j and n as in [10, Lemma 2.5]. Therefore, $H^1(K_n, T(\kappa^m))$ injects into $H^1(K_n, V(\kappa^m))$ under the natural map and we can treat the former as a lattice of the latter. In addition, the corestriction maps between $H^1(K_n, T(\kappa^m))$ are surjective and the restriction maps are injective (see [8]). We will treat $H^1(K_n, T(\kappa^m))$ as a subset of $H^1(K_{n'}, T(\kappa^m))$ for $n' \geq n$.

Let $\mathbf{z} \in \mathbb{H}^1_{\mathrm{Iw}}(T^*(1))$, then $\mathbf{z} \in \ker(\mathrm{Col}^\pm)$ iff $z_{-m,n}$ is in the annihilator of the $\mathcal{O}_F$-module generated by $\{\exp_m(\gamma_{n,m}(\xi^\pm)^\sigma) : \sigma \in G_n\}$ for all $n \geq 0$ and $0 \leq m \leq k-2$. By [10, Proposition 2.7], this is in fact equivalent to the same statement being true for all, $n \geq 0$ with one fixed $m \in \{0, \ldots, k-2\}$ (we will take $m = 0$ below).

Instead of looking at the said $\mathcal{O}_F$-module, we study the F-vector space generated by these elements inside $H^1_f(K_n, V(\kappa^m))$ first. We can then intersect it with $H^1_f(K_n, T(\kappa^m))$ to obtain the kernel.

Proposition 3.3. *The vector subspace over F of $H^1_f(K_n, V(\kappa))$ generated by the set $\{\exp(\gamma_{n,0}(\xi^\pm)^\sigma) : \sigma \in G_n\}$, is equal to*

$$\left\{x \in H^1_f(K_n, V) : \mathrm{cor}_{n/m+1} x \in H^1_f(K_m, V) \forall m \text{ even (odd)}\right\}.$$

Proof. Recall that by the proof of Lemma 1.2, we have $\sigma f(\zeta - 1) = f(\zeta^{\kappa(\sigma)} - 1)$ for any $f \in \Xi[[X]]^\psi$, $\sigma \in G_K$ and ζ a p power root of unity. Therefore, for $n > 1$

$$\sum_{\zeta^p = 1} f(\zeta\zeta_{p^n} - 1) = \mathrm{Tr}_{n/n-1}\, f(\zeta_{p^n} - 1).$$

If $n = 1$, then

$$\sum_{\zeta^p = 1} f(\zeta\zeta_p - 1) = f(0) + \mathrm{Tr}_{1/0}\, f(\zeta_p - 1).$$

Hence, we have

$$\begin{aligned}
p^n\gamma_{n,0}(\xi) &= \sum_{i=0}^{n-1}\bar{\eta}^{\varphi^{i-n}}(\zeta_{p^{n-i}}-1)\otimes\varphi^{i-n}(\xi)+\bar{\eta}(0)\otimes(1-\varphi)^{-1}(\xi)\\
&= \sum_{i=0}^{n-1}\left(\eta^{\varphi^{i-n}}(\zeta_{p^{n-i}}-1)-\frac{1}{p}\sum_{\zeta^p=1}\eta^{\varphi^{i-n}}(\zeta\zeta_{p^{n-i}}-1)\right)\otimes\varphi^{i-n}(\xi)\\
&\quad+\left(\eta(0)-\frac{1}{p}\sum_{\zeta^p=1}\eta(\zeta-1)\right)\otimes(1-\varphi)^{-1}(\xi)\\
&= \sum_{i=0}^{n}\left(\pi_{n-i}-\frac{1}{p}\operatorname{Tr}(\pi_{n-i})\right)\otimes\varphi^{i-n}(\xi)-\frac{1}{p}\operatorname{Tr}(\pi_1)\otimes(1-\varphi)^{-1}(\xi)\\
&= \sum_{i=0}^{n}\pi'_{n-i}\otimes\varphi^{i-n}(\xi)-\frac{1}{p-1}\otimes\xi+(1-\varphi)^{-1}(\xi).
\end{aligned}$$

Recall that $\varphi^2=-p^{k-3}$, so we have

$$(1-\varphi)^{-1}=\frac{1}{1+p^{k-3}}(1+\varphi).$$

In particular, $-\frac{1}{p-1}\otimes\xi^\pm+(1-\varphi)^{-1}(\xi^\pm)\notin D^0(V)$. Moreover, $\varphi^r(\omega)\in D^0(V)$ iff r is even, hence $\{\gamma_{n,0}(\xi^\pm)^\sigma\}$ generates

$$\left(K+\sum_{i\in S^\pm}K^{(i)}\right)\otimes D(V)/D^0(V)$$

where $S^\pm=\{m\in[1,n]: m \text{ even (odd)}\}$ by Corollary 3.2. Hence the result by [10, Lemma 2.8]. □

We write $H^1_f(K_n,V)^\pm$ for the vector space described in the proposition and define $H^1_f(K_n,T)^\pm=H^1_f(K_n,T)\cap H^1_f(K_n,V)^\pm$. Then,

$$H^1_f(K_n,T)^\pm=\left\{x\in H^1_f(K_n,T):\operatorname{cor}_{n/m+1}x\in H^1_f(K_m,T)\forall m \text{ even (odd)}\right\}$$

and $\ker(\mathrm{Col}^\pm)$ is given by

$$\mathbb{H}^1_{\mathrm{Iw},\pm}(T^*(1)):=\varprojlim H^1_\pm(K_n,T^*(1))$$

where $H^1_\pm(K_n,T^*(1))$ is defined to be the annihilator of $H^1_f(K_n,T)^\pm$ under the pairing

$$H^1(K_n,T^*(1))\times H^1(K_n,T)\to\mathcal{O}_F.$$

The images of $\mathrm{Col}^\pm$ can be found in the same way as [10, Section 3]. Namely, $\mathrm{Im}(\mathrm{Col}^+)\cong(u-1)\Lambda+\sum_{\sigma\in G_1}\Lambda$ and $\mathrm{Im}(\mathrm{Col}^-)\cong\Lambda$.

3.3 The even and odd Selmer groups

Let E be a number field with $[E:\mathbb{Q}]=d$. Then, the p-Selmer group of f over E is defined to be

$$\mathrm{Sel}_p(f/E)=\ker\left(H^1(E,V/T)\to\prod_v\frac{H^1(E_v,V/T)}{H^1_f(E_v,V/T)}\right)$$

where v runs through all places of E and V and T are as defined above.

Assume that p splits completely in E. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_d$ be the primes of E above p and E_∞/E a $\mathbb{Z}_p$-extension such that $\mathfrak{p}_i$ is totally ramified in E_∞. We write E_n for the nth layer. Note that $E_{\mathfrak{p}_i}$ is isomorphic to $\mathbb{Q}_p$ for $i=1,\dots,d$. By [5, Section 4.2], $E_{\infty,\mathfrak{p}_i}/E_{\mathfrak{p}_i}$ is contained in a Lubin-Tate extension for some uniformiser π of $\mathbb{Q}_p$ such that $\pi\in p(1+p\mathbb{Z}_p)$. Therefore, the $\mathrm{Col}^\pm$ restrict to $\varprojlim H^1(E_{n,\mathfrak{p}_i},T^*(1))$ and it easy to check that the description of the kernels generalise directly. For each $n\geq 0$, we can define

$$\mathrm{Sel}_p^\pm(f/E_n)=\ker\left(\mathrm{Sel}_p(f/E)\to\prod_i\frac{H^1(E_{n,\mathfrak{p}_i},V/T)}{H^1(E_{n,\mathfrak{p}_i},T)^\pm\otimes\mathbb{Q}_p/\mathbb{Z}_p}\right)$$

and $\mathrm{Sel}_p^\pm(f/E_\infty)=\varinjlim\mathrm{Sel}_p^\pm(f/E_n)$.

Unfortunately, unlike the cyclotomic case, $\mathrm{Sel}_p^\pm(f/E_\infty)$ is not Λ-cotorsion in general. However, they do satisfy a control theorem (cf [8, Theorem 9.3]) and their coranks can be used to describe those of $\mathrm{Sel}_p(f/E_n)$ (cf [5, Proposition 7.1]). Since the proofs for these results given in [5, 8] are purely algebraic and do not involve properties of elliptic curves, they generalise to general f with no difficulties.

4 Relative Lubin-Tate groups

We now assume K to be a finite unramified extension of $\mathbb{Q}_p$ of degree d. For a fixed $\pi\in\mathbb{Z}_p$ with p-adic valuation d, let g be a lift of Frobenius with respect to π in the sense of [4, Section I.1.2], then $\varphi^i(g)$ is also such a lift for any integer i. To ease notation, we will write g_i for $\varphi^i(g)$. Each g_i gives rise to an one-dimensional formal group over $\mathcal{O}_K$ which we write as $\mathcal{F}_{g_i}$. For any positive integer n, we write

$$g_i^{(n)}=\varphi^{n-1}(g_i)\circ\varphi^{n-2}(g_i)\circ\cdots\circ g_i=g_{i+n-1}\circ g_{i+n-2}\circ\cdots\circ g_i.$$

Let $W_{g_i}^n$ be the set of zeros of $g_i^{(n)}$ in $\bar{K}$ and write $K_n=K(W_{g_i}^n)$ which is independent of the choice of g and i. Moreover, if $\omega\in W_{g_i}^n\setminus W_{g_i}^{n-1}$, then $K_n=K(\omega)$. Let $\eta_i:\mathbb{G}_m\to F_{g_i}$ be an isomorphism, then $\eta_i\in\mathfrak{O}[[X]]$ and $\omega_{n,i}:=\eta_i^{\varphi^{-n}}(\zeta_{p^n}-1)\in W_{g_{i-n}}^n\setminus W_{g_{i-n}}^{n-1}$ (see [4, Section I.3.2]). Note that g_{i-n} sends $W_{g_{i-n}}^n$ to $W_{g_{i-n+1}}^{n-1}$, we define the Tate module of F_{g_i} to be

$$T_{g_i}=\varprojlim_{g_{i-n}}W_{g_{i-n}}^n.$$

Since η_i satisfies $g_i \circ \eta_i = \eta_i^{\varphi}((1+X)^p - 1)$, we have $(\omega_{n,i})_n \in T_{g_i}$.

The character κ of G_K on T_{g_i} is independent of i by [4, Proposition I.1.8]. As in the case of absolute Lubin-Tate groups, κ can be decomposed as $\kappa = \chi\psi$ where χ is the cyclotomic character and ψ is an unramified character.

Results of [15] hold in this context with the obvious modifications, especially Theorem 1.5. In particular, for any $\xi \in D(V)$ and i an integer, we can define a measure $\mu_\xi^{(i)}$ on $\mathbb{Z}_p^\times$ whose Amice transform is given by $\bar{\eta}_i(X) \otimes \xi$ where $\bar{\eta}_i$ is defined in the same way as $\bar{\eta}$ in Section 2. We can then define $\mathcal{L}_\xi^{(i)}$ as before. For $V = V_f(1)$ and $F = \mathbb{Q}_p$ (so $\mathcal{O}_F = \mathbb{Z}_p$), we define

$$\begin{array}{rcl} \mathrm{Col}^\pm : \mathbb{H}^1_{\mathrm{Iw}}(V^*(1)) & \to & \mathbb{Q} \otimes \Lambda^d \\ \mathbf{z} & \mapsto & \left(\mathcal{L}^{(i)}_{\xi^\pm}(\mathbf{z}) / \log^\pm_{p,k} \right)_{i=0,\cdots,d-1}. \end{array}$$

We now follow [7, Section 3] to find the image of Col^-. In particular, we assume that g is a polynomial of degree p and the coefficient of X^{p-1} is $\zeta_0 p$ where ζ_0 is a root of unity in K such that $\mathcal{O}_K = \mathbb{Z}_p[\zeta_0]$.

Lemma 4.1. *With the above notation,* $\left(\mathbb{E}_{h,V}(\mu^{(i)}_{\xi^-})\right)_0$, $i = 0, \cdots, d-1$, *is linearly independent over* $\mathbb{Q}_p$.

Proof. By Theorem 2.5, we have

$$\left(\mathbb{E}_{h,V}(\mu^{(i)}_{\xi^-})\right)_0 = \exp\left((1-\varphi)^{-1} \left(1 - \frac{\varphi^{-1}}{p}\right) \bar{\eta}_i(0) \otimes \xi^- \right).$$

We first simplify the expression $(1-\varphi)^{-1}(1-\frac{\varphi^{-1}}{p})$. Recall that φ satisfies

$$\varphi^2 + p^{k-3} = 0 \text{ and } (1-\varphi)^{-1} = \frac{1}{1+p^{k-3}}(1+\varphi).$$

Therefore,

$$\begin{aligned} & (1-\varphi)^{-1}\left(1 - \frac{\varphi^{-1}}{p}\right) \\ = \; & \frac{1}{1+p^{k-3}}(\varphi+1)\left(1 - \frac{\varphi^{-1}}{p}\right) \\ = \; & \frac{1}{1+p^{k-3}}\left(\varphi - \frac{\varphi^{-1}}{p} + 1 - \frac{1}{p}\right) \\ = \; & \frac{1}{1+p^{k-3}}\left(\left(1 + \frac{1}{p^{k-2}}\right)\varphi + 1 - \frac{1}{p}\right). \end{aligned}$$

We write $\lambda = (p^{2-k}+1)/(p^{k-3}+1)$. Since $\xi^- = \omega \in D^0(V)$, we have

$$(1-\varphi)^{-1}\left(1 - \frac{\varphi^{-1}}{p}\right) \bar{\eta}_i(0) \otimes \xi^- \equiv \lambda \bar{\eta}_i{}^{\varphi}(0) \otimes \varphi(\omega) \mod D^0(V).$$

But $\bar{\eta}_i{}^{\varphi}(0)$ equals to

$$\eta_i^{\varphi}(0) - \frac{1}{p} \sum_{\zeta^p=1} \eta_i^{\varphi}(\zeta - 1) = \varphi^{i+1}(\zeta_0)$$

since the summands are the roots g_i^{φ}. By definition, $\zeta_0, \varphi(\zeta_0) \cdots, \varphi^{d-1}(\zeta_0)$ is a $\mathbb{Z}_p$-basis of $\mathcal{O}_K$, so we are done. □

Corollary 4.2. *The image of* $\mathbb{H}^1_{\mathrm{Iw}}(T^*(1))$ *under* Col^- *is isomorphic to* Λ^d.

Proof. By [10, proof of Lemma 3.11], there exists an integer r such that

$$p^{-r}\left(\mathbb{E}_{h,V}(\mu_{\xi^-}^{(i)})\right)_0 \in H^1(K,T) \setminus pH^1(K,T)$$

for all i. Hence, as in [7, proof of Proposition 3.9], their linear independence over $\mathbb{Z}_p$ implies that

$$\left\{\left(p^{-r}\mathcal{L}_{\xi^-}^{(i)}(z)\right)_{i=0,\cdots,d-1} : z \in H^1(K,T)\right\} = \mathbb{Z}_p^d.$$

But the image of $\log_{p,k}^-$ in $\mathbb{Z}_p$ is a p-adic unit (see [10, Section 3.2]), so we have

$$p^{-r}\mathrm{Col}_0^-(H^1(K,T^*(1))) = \mathbb{Z}_p^d.$$

But the following diagram commutes (see [10, proof of Theorem 3.10]):

$$\begin{array}{ccccc}
H^1(K_m,T^*(1)) & \xrightarrow{p^{-r}\mathcal{L}_{\xi^-,m}^{(i)}} & \mathbb{Q}_p[G_m] & \xrightarrow{(\log_{p,k}^-)^{-1}} & \mathbb{Z}_p[G_m] \\
\downarrow{\scriptstyle \mathrm{cor}} & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\
H^1(K_n,T^*(1)) & \xrightarrow{p^{-r}\mathcal{L}_{\xi^-,n}^{(i)}} & \mathbb{Q}_p[G_n] & \xrightarrow{(\log_{p,k}^-)^{-1}} & \mathbb{Z}_p[G_n]
\end{array}$$

where $m > n$, hence the result by Nakayama's lemma. □

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