What so perfect about perfect numbers?

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The definition of perfect numbers can be found in a standard number theory textbook. A natural number n is said to be perfect if the sum of the divisors of n equals $2n$. It has been shown that an even number is perfect iff it is of the form $2^{p-1}(2^p - 1)$ where both p and $2^p - 1$ are primes. For example, 6 and 28 are perfect numbers. Whether there exist any odd perfect numbers is still an open problem.

One might ask how people came up with the idea of perfect numbers when it seems not that perfect at all. When the Egyptians first developed the idea of fractions, they only had the idea of $1/n$. If n is perfect, say with divisors, $1, n_1, ..., n_k$ and n. Then $n = 1 + n_1 + ... + n_k$. Dividing n and rearranging gives

$$1 = \frac{1}{n} + \frac{1}{n_1} + ... + \frac{1}{n_k}$$

That means 1 can be decomposed into distinct fractions. For the Egyptians, with only $1/n$ on hand, only perfect numbers have this property. Now you may ask, so? Let's not forget our ancestors didn't invent maths to prove Fermat's Last Theorem. They used maths to solve daily-life problems, like keeping track of goods, working out volumes of containers, etc. Let's see how the Egyptians solved problems.

An Egyptian just travelled to Cambridge by time machine. He has made a few friends already, they are Peter, John and Paul. He has n sheep and he wants to give them to his friends as souvenirs. Peter is his best friend and he wants to give him the most. John just bought him a drink at college bar, so he deserves more sheep than Paul. If n is a perfect number, say 6, so we have

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

Then he can give Peter 1/2 of his sheep (3 sheep), John 1/3 (2 sheep) and Paul 1/6 (1 sheep). They call it an Aliquot giving. In the same context, we can see where the definitions of excessive numbers (those larger than the sum of their factors not counting themselves) and defective numbers (those smaller than the sum of their factors not counting themselves) came from.

Now you are happy with perfect numbers. But you might ask, why on earth did the Egyptians invent $1/n$ only? Our Egyptian friend doesn't have a wallet. He just lost 8 penny coins out of 20 at formal, what fraction had he lost? Peter immediately says 2/5, but our Egyptian friend says $1/4 + 1/10 + 1/20$ (it was their custom not to use the same fraction more than once in a sum). His friends are not impressed with such a cumbersome answer, so he challenges them with the following problem. He has 6 rolls and asks them to distribute the rolls to 10 fellows evenly. Paul is good at arithmetic. He says 3/5 each. So he cuts each roll into fifths. Our Egyptian laughs at him and says, $1/2 + 1/10$. Then he divides 5 of the rolls into halves and the last roll into tenths. Each fellow gets one half plus one thenth. Surely, he finishes before Paul.

Finally, let's have a look at an extension of perfect numbers. Amicable numbers are a pair of numbers such that each number is the sum of the other's factors not counting itself. For example, 220 and 284 is such a pair. It's time for our Egyptian friend to go. He has £284 left. Since they don't use sterling in Egypt, he has decided to give his money away. He wants to give £142 to the college, £71 to the bedder, £4 to the porter, £2 to the barman at the college bar and £1 to the college cat. Oh no! He realises 284 is an excessive number and there is a surplus of £64! Peter wants to go with him because he was just dumped by his girlfriend. He has made an aliquot will as well. His bank account has £220 left and he finds that he is £64 short! Ouch, 220 is defective! So, our Egyptian friend gives him the excess £64, then everyone is happy. What a lovely friendship between amicable numbers!

References

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