Brauer Groups

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Abstract

The aim of this essay is to define the Brauer group of a field. We will start with the elementary definitions and results needed until we can define what a Brauer group is. Later, we will give some examples and more results will be proved when necessary.

1 Tensor Product

The group operation of a Brauer group is tensor product. Here, the definition of a tensor product is given and some of its properties are shown. Unless otherwise stated, all modules are left-modules.

Definition 1.1 *Let M and N be R-modules where R is a commutative ring. Let T denote the free R-module generated by elements of $M \times N$. Let V be the submodule of T generated by all elements of the form:*

$$(m + m', n) - (m, n) - (m', n)$$

$$(m, n + n') - (m, n) - (m, n')$$

$$(rm, n) - r(m, n)$$

$$(m, rn) - r(m, n)$$

Then the ***tensor product*** *of M and N (over R), denoted by $M \otimes_R N$, is defined to be the quotient module T/V.*

Theorem 1.2 (Universal Property of Tensor Product) *Let M, N and P be modules over a commutative ring R. There exists a R-bilinear map (R-linear in both coordinates) $i : M \times N \to M \otimes_R N$ s.t. given any R-bilinear map $f : M \times N \to P$, there exists a unique R-linear map $f' : M \otimes_R N \to P$ with $f = f'i$.*

Proof: Denote the image of (m, n) under the natural homomorphism from T to T/V by $m \otimes n$. Define $i : M \times N \to M \otimes_R N$ by $i(m, n) = m \otimes n$. Bilinearity of i follows from the definition of V.
Now, given $f : M \times N \to P$, we may extend it to a homomorphism $\tilde{f} : T \to P$

because $M \times N$ forms a free basis of T. Since f is R-bilinear, $\tilde{f}(V) = 0$. So, $V \subseteq \ker(\tilde{f})$. By factor theorem, there is a homomorphism (R-linear $\Leftrightarrow$ homomorphism of R-modules) $f' : T/V \to P$ s.t. $f'(m \otimes n) = \tilde{f}(m, n) = f(m, n)\ \forall\ m \in M,\ n \in N$. Also, this uniquely determines f'. But $f'(m \otimes n) = f'(i(m, n))$, hence $f'i = f$. $\Box$

Theorem 1.2 is very important and will be used over and over again in this section. It proves the next two propositions. The first one shows that a Brauer group is abelian and the second one proves the associativity of a Brauer group. From now on, we simply write $M \otimes N$ for $M \otimes_R N$ when R is clear from the context.

Proposition 1.3 $M \otimes N \cong N \otimes M$

Proof: Define $f : M \times N \to N \otimes M$ by $f(m, n) = n \otimes m$. It is clear that f is R-bilinear. Hence, by theorem 1.2, there exists a homomorphism $f' : M \otimes N \to N \otimes M$ s.t. $f'(m \otimes n) = n \otimes m$. By symmetry, there exists a homomorphism $g' : N \otimes M \to M \otimes N$ with $g'(n \otimes m) = m \otimes n$. Therefore, f' and g' are inverse of each other, hence are isomorphisms. $\Box$

Proposition 1.4 $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$

Proof: Define $f : M \times N \times P \to (M \otimes N) \otimes P$ by $f(m, n, p) = (m \otimes n) \otimes p$. Now, apply theorem 1.2 for fixed m, we have a R-bilinear map $f' : M \times (N \otimes P) \to (M \otimes N) \otimes P$ with $f'(m, (n \otimes p)) = (m \otimes n) \otimes p$. Apply theorem 1.2 to f', we have a homomorphism $f'' : M \otimes (N \otimes P) \to (M \otimes N) \otimes P$ with $f''(m \otimes (n \otimes p)) = (m \otimes n) \otimes p$. As in the last proof, we can find the inverse of f'', hence the result. $\Box$

We will now introduce a new algebraic structure, algebra. It will play an important role when we define the elements of Brauer groups.

Definition 1.5 *Let R be a commutative ring. A is an R-**algebra** if it is a ring as well as a R-module s.t. the ring and module multiplications are compatible, that is*

$$r(ab) = (ra)b = a(rb)\ \forall r \in R,\ a, b \in A$$

*The **centre** of A, denoted by $Z(A)$, is defined to be $\{a \in A : ab = ba\ \forall b \in A\}$. A is said to be **central** if $Z(A) = R$.*

Definition 1.6 *Let A and B be R-algebras. A map $f : A \to B$ is an R-**algebra homomorphism** if f is a homomorphism of R-modules as well as a ring homomorphism.*

Using tensor product, we can construct new algebras from old ones as shown by the following proposition.

Proposition 1.7 *If A and B are algebras over the commutative ring R, then $A \otimes B$ is also an R-algebra.*

Proof: Consider a map from $A \times B \times A \times B$ to $A \otimes B$ with $(a, b, a', b') \mapsto aa' \otimes bb'$. It is R-linear in each coordinate, hence, there is a homomorphism $f : A \otimes B \otimes A \otimes B \to A \otimes B$ with $f(a \otimes b \otimes a' \otimes b') = aa' \otimes bb'$ by theorem 1.2. Now, let $g : (A \otimes B) \times (A \otimes B) \to A \otimes B \otimes A \otimes B$ with $g(u, v) = u \otimes v$. Then g is R-bilinear. So, fg is a R-bilinear map with $fg(a \otimes b, a' \otimes b') = aa' \otimes bb'$ and this defines our multiplication with identity $1 \otimes 1$. It can be checked routinely that the distributive law holds.
For compatibility, note that $r[(a \otimes b)(a' \otimes b')] = r(aa' \otimes bb') = raa' \otimes bb'$, $[r(a \otimes b)](a' \otimes b') = (ra \otimes b)(a' \otimes b) = raa' \otimes bb'$ and $(a \otimes b)[r(a' \otimes b')] = (a \otimes b)(ra' \otimes b') = ara' \otimes bb' = raa' \otimes bb'$. Therefore, the two multiplications are compatible. $\square$

We will now prove a couple of results on tensor product of algebras which will be used in later sections.

Lemma 1.8 *Given a field k and finite-dimensional k-algebras A and B, let $\{a_i\}$ and $\{b_j\}$ be bases for A and B over k respectively. Then $\{a_i \otimes b_j\}$ is a basis for $A \otimes B$. In particular,* $\dim(A \otimes B) = \dim(A)\dim(B)$.

Proof: By definition, elements of $A \otimes B$ are linear combinations of elements of the form $(\sum r_i a_i) \otimes (\sum s_j b_j) = \sum r_i s_j (a_i \otimes b_j)$. Hence $\{a_i \otimes b_j\}$ forms a generating set for $A \otimes B$. For linear independence, we construct a linear functional $L : A \otimes B \to k$. Let $\{\alpha_i\}$ and $\{\beta_j\}$ be the dual bases to $\{a_i\}$ and $\{b_j\}$ respectively. For given i and j, define $f : A \times B \to k$ by $f(a, b) = \alpha_i(a)\beta_j(b)$ which is clearly k-bilinear. By theorem 1.2, we have a linear functional L s.t. $L(a \otimes b) = \alpha_i(a)\beta_j(b)$. Therefore, $L(a_i \otimes b_j) = 1$ and L vanishes on all other $a_k \otimes b_l$. Hence the linear independence. $\square$

Proposition 1.9 *Given a field k and k-algebras A and B, we have*
1. $A \otimes B$ contains A and B as commuting subalgebras.
2. Any basis $\{b_i\}$ of B over k is a basis for $A \otimes B$ as an A-module.
3. Any basis $\{a_i\}$ of A over k is a basis for $A \otimes B$ as an B-module.

Proof: Define k-algebra homomorphisms $f : A \to A \otimes B$ and $g : B \to A \otimes B$ with $f(a) = a \otimes 1$ and $g(b) = 1 \otimes b$. Let $\{b_j\}$ be a basis for B over k. Then, by lemma 1.8, for any $x \in A \otimes B$, x has a unique expression $x = \sum r_{ij}(a_i \otimes b_j) = \sum a'_j \otimes b_j$ where $a'_j = \sum r_{ij} a_i$. So, we have $x = \sum a'_j \otimes b_j = \sum (a'_j \otimes 1)(1 \otimes b_j) = \sum f(a'_j)g(b_j)$. Since the expression is unique, $\{g(b_j)\}$ forms a free basis for $A \otimes B$ when treated as an A-module. Now, $\ker(f)$ annihilates $A \otimes B$, but only 0 annihilates a free module, so $\ker(f) = \{0\}$. Therefore, f is injective. Hence, we may identify A as a subalgebra of $A \otimes B$. Now, reversing the roles of A and B, we see that the same is true for B. A and B commute since $(a \otimes 1)(1 \otimes b) = a \otimes b = (1 \otimes b)(a \otimes 1)$. $\square$

2 Simplicity

The elements of a Brauer group of a field, say k, are equivalence classes of k-algebras with certain properties. We will show how to define the equivalence relation in this section. First, we give some important definitions. They are not only important for results in this section, but also will be vital for our definition of Brauer Groups.

Definition 2.1 *A nonzero module M is **simple** if it contains no proper nonzero submodule.*

Definition 2.2 *A ring R is **simple** if it has no non-trivial two-sided ideals. An algebra A is **simple** if it is simple as a ring.*

Definition 2.3 *A ring R is a **division ring** if $\forall r \in R \setminus \{0\} \exists s \in R$ s.t. $rs = sr = 1$, ie all elements of R are invertible. A **division algebra** is an algebra which ring structure gives a division ring.*

Note that a field is just a commutative division ring. A lot of results for vector spaces are also true for modules over a division ring because we don't need commutativity when we prove them. The following is quoted without proof.

Proposition 2.4 *Any D-module is a direct sum of copies of D where D is a division ring. In particular, we can define the dimension of a D-module, as for vector spaces.*

Finally, we give the definition of an opposite algebra which will be the inverse of an element of a Brauer group.

Definition 2.5 *Let R be a ring. The **opposite ring** of R, denoted by R^{opp}, is the ring with the same additive group, but with multiplication defined by $r \cdot s = sr$. For an algebra A, the **opposite algebra** of A, denoted by A^{opp}, is the algebra with the corresponding property as a ring.*

From now on, unless otherwise stated, all algebras are over a field k of finite dimension and all tensor products are carried out over k. Note that we have already seen that A and B are finite-dimensional implies $A \otimes B$ is finite-dimensional.

Our ultimate goal of this section is to relate simple algebras to matrix algebras over division algebras. Here, $M_n(D)$ denotes the algebra of $n \times n$ matrices over D. In order to do prove the result desired, we will need a couple of lemmas.

Lemma 2.6 *Let A be an k-algebra and M be a simple A-module. Then $End_A(M)$ is a division algebra over k.*

Proof: Let $f \in End_A(M) \setminus \{0\}$. Since $\ker(f) \leq M$, by simplicity, $\ker(f) = M$ or $\{0\}$. But $f \neq 0$, so $\ker(f) = \{0\}$. Similarly, $\mathrm{im}(f) \leq M$ implies $\mathrm{im}(f) = M$. Therefore, f is an isomorphism, hence invertible. Therefore, $End_A(M)$ is a division ring. Note that k acts on M via left multiplication, so $End_A(M)$ is a k-algebra. Since the action commutes with $End_A(M)$, compatibility follows. Hence, $End_A(M)$ is a division algebra over k. □

Lemma 2.7 *With the above notation, let D be the division algebra $End_A(M)$. Then $End_A(M^n) \cong M_n(D)$ as k-algebras.*

Proof: By a similar argument as above, one may show that $End_A(M^n)$ is a k-algebra. Now, since D is a k-algebra, if we identify k with kI_n where I_n is the identity matrix, we see that $M_n(D)$ is a k-algebra as well.
Given $f \in End_A(M^n)$, let $f_{ij} \in D$ be s.t. $f_{ij}(m)$ is the ith component of $f(0, ..., m, ..., 0)$ where m is at the jth component. It is not hard to see that $f \mapsto (f_{ij})$ defines an isomorphism from $End_A(M^n)$ to $M_n(D)$. □

Theorem 2.8 *Let A be a simple algebra, then there exists a faithful simple A-module M. Moreover, $A \cong M^n$ as A-modules for some n.*

Proof: Let I be a left ideal of A, other than $\{0\}$, with the smallest dimension over k, then I is a simple A-module. Since the annihilator of M is a two-sided ideal of A, by simplicity of A, $ann(M) = A$ or $\{0\}$. But 1 is not in the annihilator, so it can only be $\{0\}$. Hence, M is a faithful simple A-module as required.
Consider all A-homomorphisms $f : A \to M^n$ for various n and choose one with minimal kernel w.r.t. inclusion (exists by finite-dimension). Assume f is not injective, then we have $f(b) = 0$ for some $b \neq 0$. We have $bm \neq 0$ for some $m \in M$ by faithfulness. Now, define $g : A \to M^n \oplus M$ with $g(a) = (f(a), am)$ and g has a smaller kernel than f (since $\ker(g) \subseteq \ker(f) \setminus \{b\}$), contradiction. Hence, f is injective. By isomorphism theorem, A is isomorphic to a submodule of M^n.
Now, for each copy of M, $A \cap M$ is a submodule of M. By simplicity of M, either $A \cap M = M$ or $\{0\}$. So, inside M^n, A is a direct sum of finitely many copies of M, hence the result. □

Lemma 2.9 *With the above notations, any simple A-modules are isomorphic to M.*

Proof: We have already seen that $A \cong M^n$ and a submodule N of $A \cong M^n$ is a direct sum of copies of M. Therefore, A/N is also a direct sum of finitely many copies of M.
Let P be a simple A-module. For $p \in P$ with $p \neq 0$, we have $\{0\} \neq Ap \leq P$. By simplicity of P, $Ap = P$. Let $f : A \to P$ with $f(a) = ap$. This is a surjective homomorphism, hence, $P \cong A/\ker(f)$ by isomorphism theorem. By the above remark, $P \cong M^m$ for some m. But P is simple, we have $m = 1$ as required. □

Corrolary 2.10 *If A is a simple k-algebra, then $A \cong M_n(D)$ for some n and a dvision k-algebra D.*

Proof: By theorem 2.8, $A \cong M^n$ as A-modules for some n. So, $End_A(A) \cong End_A(M^n)$. By lemma 2.7, $End_A(M^n) \cong M_n(D)$ where $D = End_A(M)$ is a division algebra. Hence, $End_A(A) \cong M_n(D)$.
Now, we need to simplify $End_A A$. Let $f : A \to End_A(A)$ with $f(a)(b) = ba$. So $f(ab) = f(b)f(a)$. Therefore, f is a homomorphism from A^{opp} to $End_A A$.

If $f(a) = f(b)$, then $f(a)(1) = f(b)(1)$, ie $a = b$. So, f is injective. For any $g \in End_A A$, we have $g = f(g(1))$. So f is surjective as well. Therefore, $A^{oop} \cong End_A A$.
Putting these together, we have $A^{opp} \cong M_n(D)$. It is clear that $(A^{opp})^{opp} \cong A$ and $M_n(D)^{opp} \cong M_n(D^{opp})$ (by looking at transpose). So, we can now conclude that $A \cong M_n(D^{opp})$ and D^{opp} is a division k-algebra because D is. $\Box$

The converse is also true. To prove it, we need the following lemma.

Lemma 2.11 *Let R be a ring. Then any ideals of $M_n(R)$ are of the form $M_n(I)$ where I is a two-sided ideal of R.*

Proof: Let J be a two-sided ideal of $M_n(R)$ and denote J_{ij} be the set of (i,j)-entries in J. Note that swapping columns and rows correspond to right and left multiplications by elmentary matrices respectively. Therefore, for any $1 \leq i,j,k,l \leq n$, if $r \in J_{ij}$, then $r \in J_{kl}$ as well. Hence, $J_{ij} = J_{kl}$. So, we see that J is of the form $M_n(L)$ for some subset L of R. By adding two matrices of J, we see that L is an additive subgroup. Multiply a row or a column by $r \in R$ corresponds to left or right multiplication by a matrix in $M_n(R)$. So $rL \subseteq L$ and $Lr \subseteq L$. Therefore, L is a two-sided ideal of R. $\Box$

Proposition 2.12 *$M_n(D)$ is a simple k-algebra for any divison k-algebra D.*

Proof: If I is a two sided ideal of $M_n(D)$, then $I = M_n(J)$ for some two-sided ideal J of D by lemma 2.11. Since every element of D is invertible, either $J = \{0\}$ or D. So either $I = \{0\}$ or $M_n(D)$. Therefore, $M_n(D)$ is simple. To see that it is an k-algebra, identify D with DI_n in $M_n(D)$ where I_n is the identity matrix. Since D is a k-algebra, we can identify k in the same way. Compatibility follows immediately. $\Box$

Finally, with the following theorem, we can define our equivlance relation.

Theorem 2.13 *If $M_n(D) \cong M_m(E)$ as algebras where D and E are division algebras, then $D \cong E$ and $n = m$.*

Proof: Let $A = M_n(D)$, then A is a simple algebra by proposition 2.12. Hence, there is a unique simple A-module, M, by lemma 2.9. Let I be the left ideal generated by e_{11}, the matrix with 1 at the $(1,1)$-entry and 0 everywhere else. By considering row operations (ie left multiplication), we see that I consists of those matrices which have zero entries everywhere but the first column.
Now, given any matrix $a \neq 0$ in I, using row operation, we can move a nonzero element to the first row, and eliminate all other nonzero entries if necessary. Since D is a division ring, multiplying the inverse of the first entry, we get e_{11}. Therefore, $Aa = I$. So I has no nonzero proper submodule, ie I is a simple A-module. We have $I \cong M$.
Let $f \in End_A(I)$ with $a = f(e_{11})$. For any $i \in I$, $f(i) = f(ie_{11}) = if(e_{11}) = i(a_{11}I_n)$ where a_{11} is the $(1,1)$-entry of a. We see that there is a one-one correspondence between $End_A(I)$ and D with multiplication the other way round.

Therefore, $End_A(I) \cong D^{opp}$. So, we have $D \cong (End_A(M))^{opp}$ which is uniquely determined by A. So $D \cong E$. By comparing dimensions, we see that $n = m$. $\square$

The highly anticipated equivalence relation is as follows.

Definition 2.14 *Let A and B be finite-dimensional simple k-algebras. We say A and B are **similar**, denoted by $A \sim B$, if there exist m and n s.t. $A \cong M_m(D)$ and $B \cong M_n(E)$ where D and E are isomorphic division algebras.*

It is clear that $\sim$ defines an equivalence relation on finite-dimensional simple k-algebras by corollary 2.10, theorem 2.13 and the fact that isomorphism is an equivalence relation.

3 Central Simple Algebras

The aim of this section is to prove the closure of a Brauer group. Our elements are equivalence classes of central simple algebras and the group operation is tensor product. So, we need to show that the tensor product of two central simple algebras is central simple as well.

Lemma 3.1 *Let A and B be algebras wtih B central simple. If I is a nonzero two-sided ideal of $A \otimes B$, then $I \cap A \neq \{0\}$.*

Proof: Let $x \in I$ with $x \neq 0$. We may write $x = \sum_{i=1}^{l} a_i \otimes b_i$ with l minimal. Then, $\{a_i\}$ and $\{b_i\}$ are linearly independent over k. In particular, we have $b_1 \neq 0$. Since B is simple, we have $Bb_1B = B$. Therefore, there exist x_j, $y_j \in B$ s.t. $\sum_{j=1}^{m} x_j b_1 y_j = 1$. Now, let

$$\begin{aligned} x' &= \sum_{j=1}^{m} (1 \otimes x_j) x (1 \otimes y_j) \\ &= \sum_{j=1}^{m} \sum_{i=1}^{l} a_i \otimes x_j b_i y_j \\ &= \sum_{i=1}^{l} a_i \otimes (\sum_{j=1}^{m} x_j b_i y_j) \\ &= \sum_{i=1}^{l} a_i \otimes b'_i \end{aligned}$$

where $b'_i = \sum_{j=1}^{m} x_j b_i y_j$. In particular, $b'_1 = 1$. I is a two-sided ideal, so $x' \in I$. By proposition 1.9, a_i linearly independent over k implies that a_i linearly independent over B. As $b'_1 \neq 0$, we have $x' \neq 0$.
For any $b \in B$, we have $(1 \otimes b)x' - x'(1 \otimes b) = \sum_{i=2}^{l} a_i \otimes (bb'_i - b'_i b)$ since $b'_1 = 1$. By minimality of l, this element in I has to be 0. By linear independence of a_i,

we have $bb'_i = b'_i b \ \forall b \in B$. Therefore, $b'_i \in Z(B) = k \ \forall i$ (as B is central). By compatibility, we may now conclude that

$$\begin{aligned} x' &= \sum_{i=1}^{l} a_i \otimes b'_i \\ &= \sum_{i=1}^{l} a_i b'_i \otimes 1 \\ &= (\sum_{i=1}^{l} a_i b'_i) \otimes 1 \in A \end{aligned}$$

Therefore, $x' \in I \cap A$. But $x' \neq 0$, so $I \cap A \neq \{0\}$. □

Theorem 3.2 *Let A and B be algebras with B central simple. Then, any two-sided ideal of $A \otimes B$ is of the form $I \otimes B$, where I is a two-sided ideal of A.*

Proof: Let J be a two-sided ideal of $A \otimes B$ and let $I = J \cap A$. Clearly, I is a two-sided ideal of A. Let f be the natural homomorphism from $A \otimes B$ to $(A/I) \otimes B$. If $\{x_i\}$ is a basis for I, extend it to a basis $\{x_i\} \cup \{y_j\}$ for A. Then by results from linear algebra, we see that $\{y_j + I\}$ is a basis for A/I. Hence $\sum u_i x_i + \sum v_j y_j \in \ker(f)$ iff $v_j = 0 \ \forall j$. Therefore, we may conclude that $\ker(f) = I \otimes B$. Clearly f is onto. By isomorphism theorem, we have $(A \otimes B)/(I \otimes B) \cong (A/I) \otimes B$.
If $x \in I \otimes B$, then $x = a \otimes b$ for some $a \in I = J \cap A$ and $b \in B$. So $x = (a \otimes 1)(1 \otimes b)$. But $a \otimes 1 \in J$, $x \in J$ as J is a two-sided ideal. Therefore, $I \otimes B \subseteq J$.
Assume J contains $I \otimes B$ properly. Consider the natural homomorphism $g : A \otimes B \to (A \otimes B)/(I \otimes B) \cong (A/I) \otimes B$. Then $g(J) \neq \{0\}$ and $g(J)$ is a two-sided ideal of $(A/I) \otimes B$. By lemma 3.1, $g(J) \cap A/I \neq \{0\}$. However, $I = J \cap A$ and $g(J) \cap A/(J \cap A) = \{0\}$, contradiction. Therefore, $J = I \otimes B$ as claimed. □

Theorem 3.3 *Let A and B be algebras with B central, then $Z(A \otimes B) = Z(A)$.*

Proof: By the identification of A in $A \otimes B$, it is clear that $Z(A) \subseteq Z(A \otimes B)$. Let $z = \sum a_i \otimes b_i \in Z(A \otimes B)$. As in the proof of lemma 3.1, we may assume that a_i are linearly independent over k. For $b \in B$, we have $(1 \otimes b)z = z(1 \otimes b)$. Therefore, $\sum a_i \otimes (bb_i - b_i b) = 0$. As in the proof of lemma 3.1, we have the linear independence of a_i over B. So, we can conclude that $bb_i = b_i b$, ie $b_i \in Z(B) = k$. By compatibility, we have

$$\begin{aligned} z &= \sum a_i \otimes b_i \\ &= \sum a_i b_i \otimes 1 \\ &= (\sum a_i b_i) \otimes 1 \\ &= x \otimes 1 \end{aligned}$$

where $x = \sum a_i b_i$.
For any $a \in A$, we have $(a \otimes 1)z = z(a \otimes 1)$ because $z \in Z(A \otimes B)$. Hence, $(ax - xa) \otimes 1 = 0$. So, $ax = xa \ \forall a \in A$. Therefore, $x \in Z(A)$. But $z = x \otimes 1$, so we can conclude that $Z(A \otimes B) \subseteq Z(A)$ and hence $Z(A) = Z(A \otimes B)$. $\square$

Corrolary 3.4 *If A and B are central simple algebras, then so is $A \otimes B$.*

Proof: Let J be a two-sided ideal of $A \otimes B$, then $J = I \otimes B$ for some two-sided ideal I of A by theorem 3.2. But A is simple, so $I = \{0\}$ or A. Hence $J = \{0\}$ or $A \otimes B$. So, $A \otimes B$ is simple. By theorem 3.3, $Z(A \otimes B) = Z(A)$. But A is central, so $Z(A \otimes B) = k$. Therefore, $A \otimes B$ is central simple. $\square$

4 Definition of a Brauer Group

Since our group elements are equivalence classes, we will need to show that the group operation is well-defined. The following lemma will help us prove it.

Lemma 4.1 *For any algebra A, $M_n(A) \cong A \otimes M_n(k)$.*

Proof: Let e_{ij} be the matrix with 1 at (i, j)-entry and 0 everywhere else. Then e_{ij} form a basis for $M_n(k)$ over k with the rules

$$e_{ij}e_{kl} = \delta_{jk}e_{il}, \quad \sum e_{ii} = 1 \ (*)$$

By proposition 1.9, $A \otimes M_n(k)$ is a free A-module with basis e_{ij}. By $(*)$, we see that $A \otimes M_n(k) \cong M_n(A)$. $\square$

Corrolary 4.2 *For any natural numbers m and n, $M_m(k) \otimes M_n(k) \cong M_{mn}(k)$.*

Proof: By lemma 4.1, we have $M_m(k) \otimes M_n(k) \cong M_n(M_m(k))$. Each element of $M_n(M_m(k))$ can be identified as an element of $M_{nm}(k)$ and $M_n(M_m(k)) \cong M_{nm}(k)$ by the properties of block multiplication of matrices. $\square$

Corrolary 4.3 *If $M_n(D)$ is a central simple algebra with D a division algebra, then D is central simple.*

Proof: By lemma 4.1, $M_n(D) \cong D \otimes M_n(k)$. From linear algebra, $Z(M_n(k))$ is just the set of scalar matrices as k is a field. Hence $Z(M_n(k)) = k$, ie $M_n(k)$ is central. By theorem 3.3, $Z(D \otimes M_n(k)) = Z(D)$. Since $M_n(D)$ is central, we have $Z(D) = Z(M_n(D)) = k$, ie D is central. D is trivially simple because D is a division ring. $\square$

Therefore, our equivalence relation, $\sim$, is really just a classification of central *division* k-algebras (trivially simple). Each central division algebra gives rise to one equivalence class and each equivalence class corresponds to exactly one central division algebra up to isomorphism. So why do we not consider central division algebras as our elements? The point is that the tensor product of two divison algebras may not be a division algebra. So, we consider the equivalence classes instead.

Proposition 4.4 *If $A \sim A_1$ and $B \sim B_1$, then $A \otimes B \sim A_1 \otimes B_1$.*

Proof: Let $A \cong M_n(D)$, $A_1 \cong M_{n_1}(D)$, $B \cong M_m(E)$ and $A_1 \cong M_{m_1}(E)$ where D and E are division algebras. Then we have $A \otimes B \cong M_n(D) \otimes M_m(E) = D \otimes M_n(k) \otimes E \otimes M_m(k)$ by lemma 4.1. By the commutativity of tensor product and corollary 4.2, we have $A \otimes B \cong D \otimes E \otimes M_{nm}(k)$. Apply lemma 4.1 again, we have $A \otimes B \cong M_{nm}(D \otimes E)$. Similarly, $A_1 \otimes B_1 \cong M_{n_1 m_1}(D \otimes E)$. Hence $A \otimes B \sim A_1 \otimes B_1$. $\Box$

So, our group operation $[A] \cdot [B] := [A \otimes B]$ is well-defined where $[C]$ denotes the equivalence class containing C.

Finally, we need the identity and inverse. Lemma 4.1 shows that $A \otimes k \cong k \otimes A \cong A$ (taking $n = 1$). So, we see that $[k]$ is the identity. It has been stated that A^{opp} will be our inverse. So, we need to prove $[A][A^{opp}] = [k]$. It suffices to show that $A \otimes A^{opp} = M_n(k)$ for some n as $[k] = [M_n(k)]$.

Proposition 4.5 *Let A be a central simple algebra with dimension n. Then $A \otimes A^{opp} \cong M_n(k)$.*

Proof: Let V be A regarded as a k-vector space. Let $f : A \to End_k(V)$ and $g : A^{opp} \to End_k(V)$ with $f(a)(v) = av$ and $g(a)(v) = va$. Clearly, $f(a)g(b) = g(b)f(a)$. Hence, the map $h : A \times A^{opp} \to End_k(V)$ with $h(a, b) = f(a)g(b)$ is k-bilinear. Therefore, by theorem 1.2, there is a homomorphism $i : A \otimes A^{opp} \to End_k(V)$ with $i(a \otimes b) = f(a)g(b)$. A and A^{opp} central simple implies $A \otimes A^{opp}$ central simple by corollary 3.4. By simplicity, $\ker(i) = \{0\}$ or $A \otimes A^{opp}$. But $i(1) \neq 0$, we must have $\ker(i) = \{0\}$. Therefore, i is injective.
Lemma 1.8 implies that $\dim_k(A \otimes A^{opp}) = n^2$. Since $\dim_k(V) = n$, by results from linear algebra, we have $\dim_k(End_k(V)) = n^2$. We may now conclude that i is surjective. Therefore, we have $A \otimes A^{opp} \cong End_k(V)$. Again, by results from linear algebra, we have $End_k(V) \cong M_n(k)$ which implies $A \otimes A^{opp} \cong M_n(k)$. $\Box$

We are almost there. The last thing we need to prove is that the collection of equivalence classes defined above actually forms a set.

Proposition 4.6 *Given any field k, the collection of equivalence classes of finite-dimensional central simple k-algebras defined by $\sim$ forms a set.*

Proof: As remarked above, we can consider the collection of finite-dimensional central division k-algebras (up to isomorphism), C, instead. Let A be a member of C and let $e_1, ..., e_n$ be a basis for A over k. Then $e_i e_j = \sum_{l=1}^{n} a_{ij}^l e_l$ for some $a_{ij}^l \in k$. Also, A is uniquely determinted by such a_{ij}^l. For each $n > 0$, the collection of $(a_{ij}^l)_{1 \leq i,j,l \leq n}$ form a set. So, the union of such collections over n also forms a set. Hence the result. $\Box$

Finally, here is the definition we have been waiting for.

Definition 4.7 *The* ***Brauer group*** *of a field k, denoted $Br(k)$, is the set of equivalence classes of finite-dimensional central simple k-algebras under $\sim$, with the tensor product as the group operation.*

The next theorem is really just a recap of what we have done so far.

Theorem 4.8 *$Br(k)$ is an abelian group.*

Proof: Proposition 4.6 shows that the collection of our equivalence classes actually forms a set. Corollary 3.4 shows the closure and proposition 4.4 shows that the group operation is well-defined. Proposition 1.4 proves the associativity. Lemma 4.1 gives the existence of the identity $[k]$ (by taking $n = 1$, we have $A \otimes k \cong A$). By proposition 4.5, the inverse of $[A]$ is $[A^{opp}]$. Therefore, $Br(k)$ is a group. It is an abelian group by proposition 1.3. $\square$

5 Examples of Brauer Groups

We will now give some examples. The way we do it will be quite elementary. Without advanced machinery, it will be easier to work with central division algebras. If we can find all the finite-dimensional central division algebras, $Br(k)$ can be then determined.

5.1 Algebraically Closed Fields

We will have to do some field thoery before we proceed.

Definition 5.1 *A field k is* ***algebraically closed*** *if every non-constant polynomial $f \in k[X]$ has at least one root in k.*

Note that $\mathbb{C}$ is algebraically closed by Fundamental Theorem of Algebra. If k is algebraically closed, we can factorise any $f \in k[X]$ into linear factors and this determines all the roots of f. In particular, f has at most $\deg f$ roots.

Definition 5.2 *If k and K are fields and k is a subfield of K. We say that K is an* ***extension*** *of k, written K/k. The* ***degree*** *of the extension, denoted by $[K : k]$, is $\dim_k(K)$. K/k is* ***algebraic*** *if for any element α of K, there exists a non-constant $f \in k[X]$ s.t. $f(\alpha) = 0$.*

Lemma 5.3 (Tower Law) *If we have field extensions $k \leq K \leq L$, then $[L : k] = [L : K][K : k]$*

Proof: Let $\{a_i\}$ be a basis for L over K and let $\{b_j\}$ be a basis for K over k. Given $v \in L$, there exists $\{\alpha_i\} \subseteq K$ s.t. $v = \sum \alpha_i a_i$. For each α_i, there exists $\{\beta_{ij}\} \subseteq k$ s.t. $\alpha_i = \sum \beta_{ij} b_j$. So $v = \sum \beta_{ij} b_j a_i$. Hence, $\{b_j a_i\}$ is a generating set for L over k.
For linear independence, let $\sum \gamma_{ij} b_j a_i = 0$ with $\gamma_{ij} \in k$. Then $\sum(\sum \gamma_{ij} b_j) a_i = 0$. Since $\sum \gamma_{ij} b_j \in K$, it has to be 0 by the linear independence of $\{a_i\}$. Finally, $\sum \gamma_{ij} b_j = 0$ implies $\gamma_{ij} = 0$ by the linear independence of $\{b_j\}$. Therefore, we

can conclude that $\{b_j a_i\}$ forms a basis for L over k, hence the result. $\square$

Note that we haven't used the commutativity of fields, so the same holds for modules over division rings.

Definition 5.4 *K is an* ***algebraic closure*** *of k if K/k is algebraic and K is algebraiclly closed.*

The next result from field theory will be quoted without proof.

Theorem 5.5 *For any field k, there is a unique algebraic closure up to isomorphism.*

We now return to our investigation of central division algebras.

Proposition 5.6 *Let k be an algebraically closed field. If D is a finite dimensional division k-algebra, then $k = D$.*

Proof: It is clear that $k \subseteq D$. Let $n = \dim_k(D)$ and $x \in D$. Then $1, x, x^2, ..., x^n$ are linearly dependent. There exist $a_0, a_1, ..., a_n$ not all 0 s.t. $a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = 0$. Therefore, there exists a non-constant $f \in k[X]$ s.t. $f(x) = 0$. But k is algebraically closed, so $x \in k$. Hence $D \subseteq k$ and we can conclude that $k = D$. $\square$

Corrolary 5.7 *If k is algebraically closed, $Br(k)$ is the trivial group.*

Proof: By proposition 5.6, the only finite-dimensional division k-algebra is k. Hence, there is only one equivalence class in $Br(k)$, namely $[k]$. Therefore, $Br(k)$ is the trivial group. $\square$

In particular, we see that $Br(\mathbb{C})$ is the trivial group because $\mathbb{C}$ is algebraically closed.

5.2 The Skolem-Noether Theorem

Our next goal is to find the Brauer group of a finite field and $Br(\mathbb{R})$. In order to do that, we will need the Skolem-Noether Theorem and the Centraliser Theorem which we will prove in this section and the next section respectively. To prove Skolem-Noether, we will need the next lemma.

Lemma 5.8 *Let A be a finite-dimensional simple algebra over k. If M_1 and M_2 are finitely generated A-modules of the same dimension over k, then $M_1 \cong M_2$.*

Proof: By theorem 2.8 and lemma 2.9, $A \cong M^n$ for some unique (up to isomorphism) simple A-module M. For any finitely generated A-module N, $N \cong A^m/B$ for some m and submodule B of A^m. But A^m is a direct sum of copies of M, so the same must hold for B as M is simple. Therefore, N is just a direct sum of copies of M. So, we have $M_1 \cong M^{l_1}$ and $M_2 \cong M^{l_2}$ for some l_1 and l_2. Therefore, we have $\dim_k(M_i) = l_i \dim_k(M)$ for $i = 1, 2$. But $\dim_k(M_1) = \dim_k(M_2)$, so $l_1 = l_2$. Hence $M_1 \cong M_2$. $\square$

Definition 5.9 *An automorphism α is **inner** if it is of the form $\alpha(x) = zxz^{-1}$ for a fixed element z. In this case, we will write α_z for α.*

Theorem 5.10 (Skolem-Noether) *Let A be a simple k-algebra and let B be a finite-dimensional central simple k-algebra. If $f, g : A \to B$ are homomorphisms, then there is an inner automorphism $\alpha : B \to B$ s.t. $\alpha f = g$.*

Proof: By corollary 2.10, $B \cong M_n(D)$, some division k-algebra D. By lemma 2.7, $End_D(D^n) \cong M_n(End_D D)$ because D is a simple D-algebra. As we have seen in the proof of corollary 2.10, $End_C(C) \cong C^{opp}$ for any division algebra C. So $End_C(C^n) \cong M_n(C^{opp})$. Let $C = D^{opp}$, we have $B \cong M_n(D) \cong End_E(E^n)$ where $E = D^{opp}$ is a division k-algebra. By corollary 4.3, $Z(B) = Z(D) = Z(E) = k$.
Now, note that f and g define actions of A on E^n via $a \cdot_f v := f(a)(v)$ and $a \cdot_g v := g(a)(v)$ respectively as $f(a)$, $g(a) \in End_E(E^n)$. So, we have two A-module structure on E^n, say M_f and M_g. It is clear that they have the same dimension over k. So, by lemma 5.8, they are isomorphic. Hence, there is an isomorphism from M_f to M_g. That is, $h(a \cdot_f v) = a \cdot_g h(v)\ \forall v \in E^n$ for some $h \in B$. Therefore, we have $h(f(a)v) = g(a)h(v)$. So, $hf(a) = g(a)h$ or $hf(a)h^{-1} = g(a)$, ie $\alpha_h f = g$. $\square$

5.3 The Centraliser Theorem

We need to introduce the concept of centraliser so that we can prove Wedderburn's Theorem and Frobenius' Theorem which will give us the Brauer group of a finite field and $Br(\mathbb{R})$ respectively in the end.

Definition 5.11 *If A is an algebra and B is any subset of A, the **centraliser** of B in A is defined to be $C(B) = \{a \in A : ab = ba\ \forall b \in B\}$.*

It is easy to check that $C(B)$ is a subalgebra of A. We will now prove the centraliser theorem.

Theorem 5.12 (Centraliser Theorem) *Let A be a simple subalgebra of B where B is a finite dimensional central simple k-algebra. Let D_1 and D_2 be division algebras. Then*
1. $C(A)$ is simple.
2. If $B \sim D_1$ and $A \otimes D_1^{opp} \sim D_2$, then $C(A) \sim D_2^{opp}$.
3. $[B : k] = [A : k][C(A) : k]$.
4. $C(C(A)) = A$.

Proof: As in the proof of theorem 5.10, $B \cong End_D(D^n) \cong M_n(D^{opp})$ for some division algebra D with centre k. Also, A, as an subalgebra of B, acts on D^n, hence D^n is an $A \otimes D$-module. By the definition of endomorphism, we see that $C(A) = End_{A\otimes D}(D^n)$.
1. D is central simple, so $A \otimes D$ is simple by theorem 3.2. There exists N, the unique simple $A \otimes D$-module by lemma 2.9 and $E = End_{A\otimes D}(N)$ is a

division algebra by lemma 2.6. As we have seen in the proof of lemma 2.9, any $A \otimes D$-module is of the form N^m. Hence, $D^n \cong N^m$ for some m. Hence,

$$\begin{aligned} C(A) &= End_{A\otimes D}(N^m) \\ &\cong M_m(End_{A\otimes D}(N)) \text{ by lemma 2.7} \\ &\cong M_m(E) \end{aligned}$$

$M_n(E)$ is simple by proposition 2.12, ie $C(A)$ is simple.
2. $B \cong M_n(D^{opp})$, so $D_1 \cong D^{opp}$. Since $(D^{opp})^{opp} \cong D$, we have $D_2 \sim A \otimes D_1^{opp} \cong A \otimes D$. Since E is the unique simple $A \otimes D$-module, $A \otimes D \sim E^{opp}$ as we have seen in corollary 2.10. So, $E^{opp} \cong D_2$. But $C(A) \cong M_n(E)$, so $C(A) \sim E \cong D_2^{opp}$ as required.
3. $C(A) \cong M_m(E)$, so by the tower law, $[C(A):k] = [M_m(E):E][E:k] = m^2[E:k]$. Also, $D^n \cong N^m$, we have $[D^n:k] = [D^n:N][N:k] = m[N:k] = m[N:E][E:k]$. Cancelling m, we have

$$\begin{aligned} [C(A):k] &= \Big(\frac{[D^n:k]}{[N:E][E:k]}\Big)^2[E:k] \\ &= \frac{[D^n:k]^2}{[N:E]^2[E:k]} \\ &= \frac{[D^n:k]^2}{[End_E(N):E][E:k]} \text{ as } E \text{ is a division algebra} \\ &= \frac{[D^n:k]^2}{[End_E(N):k]} \\ &= \frac{[D^n:k]^2}{[A\otimes D:k]} \\ &= \frac{[D^n:k]^2}{[A:k][D:k]} \text{ by lemma 1.8} \end{aligned}$$

So, simplifying the above equation, we have

$$\begin{aligned} [A:k][C(A):k] &= \frac{[D^n:k]^2}{[D:k]} \\ &= \frac{([D^n:D][D:k])^2}{[D:k]} \\ &= n^2[D:k] \\ &= [M_n(D):k] \\ &= [B:k] \text{ as } B \cong M_n(D^{opp}) \end{aligned}$$

4. It is clear that $A \subseteq C(C(A))$. Apply part 3 to $C(A)$, we have $[B:k] = [C(A):k][C(C(A)):k]$. But we also have $[B:k] = [A:k][C(A):k]$, so cancelling gives $[C(C(A)):k] = [A:k]$. By results from linear algebra, we have $A = C(C(A))$. $\square$

Corrolary 5.13 *If A is a central simple subalgebra of a finite-dimensional central simple algebra B, then $B \cong A \otimes C(A)$.*

Proof: As in the proof of proposition 4.5, there is a homomorphism $f : A \otimes C(A) \to B$ with $f(a \otimes a') = aa'$ because A and $C(A)$ commute. $A \otimes C(A)$ is simple by lemma 3.1. By simplicity, either $\ker(f) = \{0\}$ or $\ker(f) = A \otimes C(A)$. But $1 \otimes 1 \notin \ker(f)$, so $\ker(f) = \{0\}$ and f is injective. Therefore, $A \otimes C(A)$ is isomorphic to a subalgebra of B. By part 3 of the Centraliser Theorem and lemma 1.8, we have $[B : k] = [A : k][C(A) : k] = [A \otimes C(A) : k]$. Hence, $B \cong A \otimes C(A)$. □

Corrolary 5.14 *Let D be a division algebra with centre k and $[D : k] = n^2$. If K is a maximal subfield of D (w.r.t. inclusion), then $[K : k] = n$.*

Proof: $k = Z(D)$ is a field because it is commutative and is a subalgebra of a division algebra. By the commutativity of K, $K \subseteq C(K)$. Given $a \in C(K)$, $K(a)$ (the minimal field containing K and a) is a subfield of D and $K \subseteq K(a)$. By the maximality of K, we have $K = K(a)$. Hence, $a \in K$ and we can conclude that $C(K) \subseteq K$. Therefore, $K = C(K)$. By part 3 of the Centraliser Theorem, $n^2 = [D : k] = [K : k][C(K) : k] = [K : k]^2$. Hence the result. □

The above corollary assumes that $[D : k]$ is a perfect square. We will see that this is always the case.

Lemma 5.15 *Let K/k be a field extension and let A be an k-algebra. Then $K \otimes_k A$ is a K-algebra, denoted by A_K. Moreover, if $\{a_i\}$ is a basis for A over k, then it is a basis for A_K over K. In particular, $[A : k] = [A_K : K]$.*

Proof: By proposition 1.9, A_K is a K-module with basis $\{a_i\}$. A_K is a ring because it is an k-algebra by proposition 1.7. Compatibility follows easily because K is a vector space over k. □

Theorem 5.16 *If D is a finite-dimensional division algebra over its centre k, then $[D : k]$ is a perfect square.*

Proof: Let K be the algebraic closure of k. By lemma 5.15, $[D : k] = [D_K : K]$, so D_K is finite-dimensional over K. By lemma 3.2, $D_K = D \otimes k$ is simple because D is central simple. Therefore, $D_K \cong M_n(E)$ for some division K-algebra E by corollary 2.10. K is algebraically closed implies that $E = K$ by proposition 5.6. So, $[D : k] = [D_K : K] = [M_n(K) : K] = n^2$ as claiemd. □

5.4 Finite Fields

Throughout this section, we assume k is a finite field. We will show that $Br(k)$ is again the trivial group. To show it, we will need Wedderburn's Theorem. First, we give a lemma from group theory.

Lemma 5.17 *If $H \leq G$ are finite groups with $H \neq G$, then $G \neq \cup_{g \in G}(gHg^{-1})$.*

Proof: Since $[G : N_G(H)]$ is the number of subgroups in G which are conjugate to H, the number of non-identity elements in $\cup_{g\in G}(gHg^{-1})$ is

$$\begin{aligned}
&\leq [G : N_G(H)](|H| - 1) \\
&\leq [G : H](|H| - 1) \text{ since } H \subseteq N_G(H) \\
&= \frac{|G|}{|H|}(|H| - 1) \\
&= |G| - \frac{|G|}{|H|} \\
&< |G| - 1 \text{ as } H \neq G
\end{aligned}$$

Therefore, $G \neq \cup_{g\in G}(gHg^{-1})$ as required. □

Theorem 5.18 (Wedderburn) *Any finite division ring is commutative.*

Proof: Let D be a division ring and let $k = Z(D)$ be a subfield of D. Let K be a maximal subfield of D containing k, so $k \subseteq K \subseteq D$. If $K = D$, then D is a field, hence commutative. So assume $K \neq D$. By theorem 5.16 and corollary 5.14, we have $[D : k] = n^2$ and $[K : k] = n$ for some n. Therefore, $|K| = |k|^n$. Quoting from field theory, any fields containing k with order $|k|^n$ are isomorphic, hence conjugate by the Skolem-Noether Theorem. So, we have $K' = zKz^{-1}$ for any maximal subfield K'.
Note that every element of D is contained in some maximal subfield, we have $D = \cup_{z\in D}(zKz^{-1})$. Now, if we take the multiplicative group, we have $D^* = \cup_{z\in D}(zK^*z^{-1})$. But this contradicts lemma 5.17. Therefore, the assumption that $K \neq D$ is false. Hence $K = D$ and D is commutative. □

Corrolary 5.19 *$Br(k)$ is the trivial group.*

Proof: Let D be a central division k-algebra. Then D is commutative by Wedderburn's Theorem. D is central, so $Z(D) = k = D$. Therefore, there is only one element in $Br(k)$, that is $[k]$. Hence the result. □

5.5 Br($\mathbb{R}$)

All our examples so far give the trivial group. We will see that this is not so for the reals. First, let's recall what the Quaternions are.

Definition 5.20 *The Quaternions, denoted by $\mathbb{H}$, is the four-dimensional vector space over $\mathbb{R}$ with basis $\{1, i, j, k\}$, and the multiplication is defined so that 1 is the multiplicative identity and*

$$\begin{aligned}
&i^2 = j^2 = k^2 = -1 \\
&ij = -ji = k \\
&jk = -kj = i \\
&ki = -ik = j.
\end{aligned}$$

It can be checked routinely that $\mathbb{H}$ is an $\mathbb{R}$-algebra. For any $q = a+bi+cj+dk \in \mathbb{H}$ and $q \neq 0$ with $a, b, c, d \in \mathbb{R}$, it can be verified that $(a - bi - cj - dk)/(a^2 + b^2 + c^2 + d^2)$ is the multiplicative inverse of q. Therefore, $\mathbb{H}$ is in fact a division $\mathbb{R}$-algebra.

Lemma 5.21 *The only finite field extensions of* $\mathbb{R}$ *are* $\mathbb{R}$ *and* $\mathbb{C}$.

Proof: Let K be a finite extension of $\mathbb{R}$ of dimension say n. Given $x \in K$, $1, x, x^2, ..., x^n$ are linearly dependent over $\mathbb{R}$. So, there exist $a_0, a_1, ..., a_n$ not all zero s.t. $a_n x^n + ... + a_1 x + a_0 = 0$. Since $\mathbb{C}$ is algebraically closed by Fundamental Theorem of Algebra, $x \in \mathbb{C}$. Therefore, $K \subseteq \mathbb{C}$.
If $x \in \mathbb{R}\ \forall x \in K$, then $K = \mathbb{R}$. If $x \in \mathbb{C} \setminus \mathbb{R}$ for some $x \in K$, say $x = a + bi$ where $b \neq 0$. $i = (x - a)/b \in K$, hence $\mathbb{C} \subseteq K$. We can then conclude that $\mathbb{C} = K$. $\square$

Theorem 5.22 (Frobenius) *If D is a division algebra with $\mathbb{R}$ in its centre and $[D : \mathbb{R}] < \infty$, then $D = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Proof: Let K be a maximal subfield of D, then $[K : \mathbb{R}] \leq [D : \mathbb{R}] < \infty$. By lemma 5.21, either $K = \mathbb{R}$ or $K = \mathbb{C}$. Assume $K = \mathbb{R}$. Note that $\mathbb{R} \subseteq Z(D)$, by the maximality of K, $Z(D) = \mathbb{R}$ and D is central. By corollary 5.14, $[D : \mathbb{R}] = 1$, hence $D = \mathbb{R}$.
If $K = \mathbb{C}$, either $Z(D) = \mathbb{C}$ or $Z(D) = \mathbb{R}$ because $\mathbb{C}$ is a maximal subfield containing $\mathbb{R}$ and $Z(D)$ is a field containing $\mathbb{R}$. If $Z(D) = \mathbb{C}$, then D is a central $\mathbb{C}$-algebra because $\mathbb{C}$ is contained in D. By corollary 5.14, $[D : \mathbb{C}] = 1$ because $[K : \mathbb{C}] = 1$. Hence $D = \mathbb{C}$.
Finally, for the case $K = \mathbb{C}$ and $Z(D) = \mathbb{R}$, D is central and we can again apply corollary 5.14 to get $[D : \mathbb{R}] = [\mathbb{C} : \mathbb{R}]^2 = 4$. Let $f : \mathbb{C} \to \mathbb{C}$ with $f(a+bi) = a-bi$ where $a, b \in \mathbb{R}$. Then f is an $\mathbb{R}$-isomorphism. By the Skolem-Noether Theorem, there exists $x \in D$ with $x \neq 0$ s.t. $x(a + bi)x^{-1} = a - bi$. Now, apply the conjugation twice, we have $x^2(a + bi)x^{-2} = a + bi$ and hence $x^2(a + bi) = (a + bi)x^2$. So, $x^2 \in C(\mathbb{C}) = \mathbb{C}$. We may apply f to x^2 and get $f(x^2) = x^2$. Hence the imaginary part of x^2 has to be 0 and $x^2 \in \mathbb{R}$. It is clear that $x^2 < 0$ otherwise $x \in \mathbb{R}$ which is not possible. So there exists $y \in \mathbb{R}\setminus\{0\}$ s.t. $x^2 = -y^2$. Let $j = x/y$ and let $k = ij$. It can be checked that the multiplication of $1, i, j, k$ here conincides with that of $\mathbb{H}$. Since $[D : \mathbb{R}] = 4$, we have $D = \mathbb{H}$. $\square$

Corrolary 5.23 $Br(\mathbb{R}) \cong \mathbb{Z}_2$.

Proof: Let D be a finite-dimensional central division $\mathbb{R}$-algebra. D can only be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. But $\mathbb{C}$ is not central, so D can only be $\mathbb{R}$ or $\mathbb{H}$. It is not hard to check that $\mathbb{H}$ is central. So, we see that $Br(\mathbb{R})$ has exactly two elements and $Br(\mathbb{R}) \cong \mathbb{Z}_2$. $\square$

Note that the identity of the group is $[\mathbb{R}]$ and the non-identity element is $[\mathbb{H}]$. So, we have $[\mathbb{H}] \cdot [\mathbb{H}] = [\mathbb{R}]$, ie $\mathbb{H} \otimes \mathbb{H} \cong M_n(\mathbb{R})$ for some n. The dimension works out to be 16, hence $n = 4$. We have $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$. We can also conclude that every finite-dimensional central simple algebra over $\mathbb{R}$ is isomorphic to a matrix algebra over $\mathbb{R}$ or $\mathbb{H}$.

Epilogue

We have defined what a Brauer group is and we have seen that it is a classifier of finite-dimensional central division algebras. This is how we have found our examples - to find all such algebras. It is not possible to do so for a general field. The connection of Brauer groups and Galois Theory leads to the computation of Brauer groups using relative Brauer groups. One can also use theory of cohomology to show that a Brauer group is torsion.

Brauer groups also have vast application in number theory, algebraic geometry, representation theory and algebraic K-theory. In particular, the study of Brauer groups of algebraic number fields has shown that there is a strong connection between Brauer groups and crossed product algebras. So, what we will have done here is really just the end of the beginning.

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