Coleman-adapted Rubin-Stark Kolyvagin systems and supersingular Iwasawa theory of CM abelian varieties

KÂZIM BÜYÜKBODUK AND ANTONIO LEI

ABSTRACT. The goal of this article is to study the Iwasawa theory of an abelian variety A that has complex multiplication by a CM field F that contains the reflex field of A, which has supersingular reduction at every prime above p. To do so, we make use of the signed Coleman maps constructed in our companion article [BL14] to introduce signed Selmer groups as well as a signed p-adic L-function via a reciprocity conjecture we formulate for the (conjectural) Rubin-Stark elements (which is a natural extension of the reciprocity conjecture for elliptic units). We then prove a *signed main conjecture* relating these two objects. To achieve this, we develop along the way a theory of Coleman-adapted rank-g Euler-Kolyvagin systems to be applied with Rubin-Stark elements and deduce the main conjecture for the maximal $\mathbb{Z}_p$-power extension of F for the primes failing the ordinary hypothesis of Katz.

CONTENTS

2000 *Mathematics Subject Classification.* 11G05; 11G07; 11G40; 11R23; 14G10.
Key words and phrases. Iwasawa Theory, Abelian Varieties, Rubin-Stark elements, Higher rank Euler systems, Kolyvagin systems.

1. Introduction

Let F be a CM field and suppose $[F:\mathbb{Q}]=2g$ and $\mathfrak{d}=\mathfrak{d}(F)$ denote Leopoldt's defect[1]. Let K denote the maximal totally real subfield of F. We fix forever a prime $p>3$ that is unramified in F. Iwasawa theory over the CM field F has been studied in [Hid06, Hid09, Hsi14, Mai08, Büy13b] under a certain p-ordinary hypothesis of Katz. Our goal here is to carry out a similar task in the absence of this hypothesis; as a matter of fact, in some sense in the other extreme case when:

(H.K.) Every prime of F above p is of degree 2 over the prime of K that lies below it.

That we work under the hypothesis (H.K.) is mostly due to our effort to keep the length of our exposition within reasonable limits. See Remark 1.3 below for a discussion regarding this point.

More precisely, what we establish in this article is as follows:

(i) We prove a $(g+1)$-variable main conjecture for a CM field F using what we call the (conjectural) *Perrin-Riou-Stark elements*[2];
(ii) We prove a divisibility in the (one-variable) *cyclotomic* main conjecture for a p-supersingular abelian variety that has CM by F and defined over K, as formulated in [BL14] (still assuming the Perrin-Riou-Stark conjecture[3]) ;
(iii) We apply (i) in order to deduce that the Perrin-Riou-Stark Kolyvagin systems utilized to prove (ii) are primitive (in the sense of [MR04, Definition 5.3.9]). The bound these Kolyvagin systems give is therefore sharp by a suitable extension of [MR04, Theorem 5.3.10(iii)] (which we discuss as part of Theorem A.14).

The step (iii) concludes the proof of the *signed main conjecture* for an abelian variety that has CM by F (or equivalently, Perrin-Riou's main conjecture in this context).

The proof of (i) and (ii) is similar to a two-variable main-conjecture and signed main conjecture proved in a recent work of the first named author [Büy13c] for a CM elliptic

[1] As we will be assuming Leopoldt's conjecture for all the main results of this article, $\mathfrak{d}$ will be zero in the statements of our main results.

[2] These are essentially the (conjectural) Rubin-Stark elements along the maximal $\mathbb{Z}_p$-power extension F_∞ of F. Precise definition of these elements is given in Section 4.2. Their defining property is inspired from Perrin-Riou's notion of higher rank Euler systems, see Remark 4.15 for a comparison of what we call the Perrin-Riou-Stark conjecture to the original Rubin-Stark conjecture.

[3] One may in fact deduce (i) only utilizing the Rubin-Stark elements. However, our sights are set on the cyclotomic main conjecture for a CM abelian variety and for this portion we make use of the Perrin-Riou-Stark elements, that (by definition) enjoy a slightly stronger norm-compatibility along F_∞/F.

curve defined over a general totally real field, modulo the complications that arise in the current work due to the fact that the images of the *signed Coleman maps* that we have constructed in our companion article [BL14] are not necessarily free. We overcome this technical difficulty in Appendix A below.

The proof of (iii) relies on various constructions we carry out in this article; essential ingredient being the rigidity of Kolyvagin systems and the signed Coleman maps again. The former point is one of the novelties in this article. In [PR04], the corresponding statement (Kobayashi's conjecture) was deduced from a (two-variable) CM main conjecture by a descent argument. For this reason, Pollack and Rubin had to utilize the non-existence of pseudo-null submodules of various Iwasawa modules. The analogous statements are not available in our context and our methods in Section 7 here are designed exactly to by-pass this issue.

We advise the reader that all these steps are listed in concrete form in the statement Theorem 7.7.

Before we explain our results in greater detail, we set some notation. Let $A_{/F}$ be a principally polarized abelian variety which has CM by F. Fix once and for all an odd prime p that is is unramified in F and is such that the endomorphism ring $\mathrm{End}_F(A)$ is an order in F whose index inside the maximal order is coprime to p. We assume further that the field F contains the reflex field of (the CM pair (F,Σ_A) associated to) the CM abelian variety A. Let Φ be the completion of F at a prime $\mathfrak{p}$ above p and let $\mathfrak{O}$ denote its ring of integers.

Let F_∞ denote the unique $\mathbb{Z}_p^{g+1+\mathfrak{d}}$-extension of F and F^{cyc} the cyclotomic $\mathbb{Z}_p$-extension. Let $\Gamma = \mathrm{Gal}(F_\infty/F)$, $\Gamma_{\mathrm{cyc}} = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$ and γ_{cyc} be a fixed topological generator of Γ_{cyc}. We define the $(g+1+\mathfrak{d})$-variable (resp., one-variable) Iwasawa algebra $\Lambda := \mathfrak{O}[[\Gamma]]$ (resp., $\Lambda_{\mathrm{cyc}} := \mathfrak{O}[[\Gamma_{\mathrm{cyc}}]]$).

1.1. **Statements of the results.** The principal objective in this article is to study the cyclotomic Iwasawa theory of the abelian variety A/F for supersingular primes. Let $A^\vee$ denote the dual abelian variety and let $I \subset \{1,\cdots,2g\}$ be a subset which is chosen so as to verify the conclusion of Proposition 6.9. Fix a prime $\mathfrak{p}$ of F above p. The starting point of our strategy involves introducing *signed local conditions* (as in Section 5)

$$H^1_I\left(F(\boldsymbol{\mu}_{p^\infty})_p, A^\vee[\mathfrak{p}^\infty]\right) \subset H^1\left(F(\boldsymbol{\mu}_{p^\infty})_p, A^\vee[\mathfrak{p}^\infty]\right)$$

in the spirit of Kobayashi [Kob03] with the aid of the Coleman maps we defined in [BL14]. The signed-subgroups we have introduced are in turn used to define a signed-Selmer group $\mathrm{Sel}^I_{\mathfrak{p}}(A^\vee/F^{\mathrm{cyc}})$ (that we will prove to be cotorsion under certain hypotheses, as well as that they *control* the classical Selmer group, see Proposition 8.3 below).

We next formulate a reciprocity law for the conjectural Perrin-Riou-Stark elements (Conjecture 4.18 below) much in the spirit of the explicit reciprocity law for elliptic units. Assuming the truth of this conjecture and using the signed Coleman maps we introduce in §7.2 the signed p-adic L-functions $\mathcal{L}^I_{\mathfrak{p}}(A^\vee) \in \Lambda_{\mathrm{cyc}}$ which satisfy a suitable interpolation property (see Proposition 7.15).

We have the following theorem concerning the *signed main conjecture* which compares the signed Selmer group to the relevant signed p-adic L-function, generalizing the main conjecture proved in [PR04]. *Assume the truth of Rubin-Stark conjectures for pro-p abelian extensions of F as well as its strengthening (which we called Perrin-Riou-Stark* *conjecture in the main text) Conjecture 4.14 and Leopoldt's conjecture for the field $F(A[\mathfrak{p}])$* for Theorems A, B and C below.

Remark 1.1. Although we have no way to verify Perrin-Riou's conjecture (which we have to assume for our main results), we are able to present a modest evidence towards its truth in Appendix B. More precisely, we are able to prove that the Kolyvagin systems that the Perrin-Riou-Stark elements ought to produce do exist unconditionally over the maximal $\mathbb{Z}_p$-tower.

Theorem A (Theorem 7.16)**.** *If the Perrin-Riou-Stark elements verify the Explicit Reciprocity Conjecture 4.18, then*

$$\mathrm{char}\left(\mathrm{Sel}_{\mathfrak{p}}^{I}(A^{\vee}/F^{\mathrm{cyc}})^{\vee}\right)=\frac{\mathcal{L}_{\mathfrak{p}}^{I}(A^{\vee})}{(\gamma_{\mathrm{cyc}}-1)^{n(I)}}\cdot\Lambda_{\mathrm{cyc}}\,,$$

where $n(I)\geq 0$ is some integer determined by the image our signed Coleman map.

Note that the constant $n(I)$ is in fact 0 when we choose an appropriate basis of the Dieudonné module of the abelian variety at p. Theorem A would have the following consequences towards the Birch and Swinnerton-Dyer conjecture for the CM abelian variety A over F. For $\alpha,\beta\in\overline{\mathbb{Q}}_p$ we write $\alpha\sim_p\beta$ if $\mathrm{ord}_p(\alpha)=\mathrm{ord}_p(\beta)$. The periods $\Omega^{J}_{\psi,p}$ and Ω^{J}_{ψ} are introduced as part of the (conjectural) description of Perrin-Riou's p-adic L-function in Conjecture 4.18, whereas the coefficients $\mathcal{D}_{I,J}$ in Definition 7.14.

Theorem B (Theorem 8.4)**.** *Assume that the hypotheses of Theorem A hold true and let $I\in\mathfrak{J}$ be chosen as above with $n(I)=0$. The following two assertions are equivalent:*

1. $L_{\{p\}}(\psi,1)\neq 0$ *and the p-adic period* $\sum_{J\in\mathfrak{J}}\mathcal{D}_{I,J}\frac{\Omega^{J}_{\psi,p}}{\Omega^{J}_{\psi}}$ *does not vanish.*
2. *The $\mathfrak{p}$-adic Selmer group $\mathrm{Sel}_{\mathfrak{p}}(A^{\vee}/F)$ of the dual abelian variety $A^{\vee}$ is finite.*

In either case,

$$|\mathrm{Sel}_{\mathfrak{p}}(A^{\vee}/F)|\sim_p L_{\{p\}}(\psi,1)\cdot\sum_{J\in\mathfrak{J}}\mathcal{D}_{I,J}\frac{\Omega^{J}_{\psi,p}}{\Omega^{J}_{\psi}}\quad.$$

The sharpness we were able to achieve both in Theorems A and B is thanks to our result on the *the (g+1)-variable supersingular Iwasawa main conjecture for F_∞/F* (Theorem C below). Theorem C is a generalization of the two-variable main conjecture proved in [Rub91, Büy13c] for certain class of Dirichlet characters of F. Before we can state it we need to introduce further notation.

Let ρ denote the Dirichlet character of F giving the action of G_F on $A[\mathfrak{p}]$ (given as in Definition 2.1 and denoted by ω_ψ when we would like to emphasize its relation with a certain Hecke character ψ associated to A). Let $\widetilde{\mathfrak{C}}^{\rho}$ denote a certain signed Coleman map (given as in Definition 5.13) and let $\mathrm{loc}_p^{\otimes g}(\varepsilon^{\rho}_{F_\infty})$ denote the image of the tower of Rubin-Stark elements (defined as in Definition 4.6) under the semi-localization map at p.

Let $\widehat{X}_\infty$ be a certain Iwasawa module (denoted by $H^1_{\mathcal{F}_I^*}(F,\mathbb{T}_\rho^*)^{\vee}$ in the main text); see Section 6 for a precise definition of this Λ-module.

Theorem C (Theorem 7.7)**.** *The element $\widetilde{\mathfrak{C}}^{\rho}\circ\mathrm{loc}_p^{\otimes g}(\varepsilon^{\rho}_{F_\infty})$ generates the ideal* $\mathrm{char}(\widehat{X}_\infty)$.

Theorem C was proved by Rubin in [Rub91, §11] when $g = 1$ using the elliptic unit Euler system. Our result, which holds true for any g, uses the Perrin-Riou-Stark elements[4] and the Coleman-adapted rank g Euler-Kolyvagin system machinery developed in Appendix A by refining some of the results of [Büy13c, Büy10].

Remark 1.2. When F verifies the p-ordinary hypothesis of Katz, a multi-variable main conjecture for F may be proved unconditionally in a wide-variety of cases relying on the Eisenstein/CM ideal method, c.f. [Hid06, Hid09, Mai08, Hsi14]. When the p-ordinary hypothesis fails, however, this approach breaks down. For this reason we still hope that the results we present above towards the CM main conjecture, albeit being conditional on the truth of the Perrin-Riou-Stark conjecture, will shed some light on the Iwasawa theoretic study of general CM fields.

Note however that we are able to prove in Appendix B that the Kolyvagin systems which the conjectural Perrin-Riou-Stark elements yield do in fact exist *unconditionally*. It would be very interesting to make use of this fact in order to deduce unconditional versions of Theorems 7.7, 7.16 and 8.4 in certain situations, such as when the CM field F is absolutely abelian.

Remark 1.3. We have stated our results when the Katz' p-ordinary hypothesis fails in the most extreme way and the prime p verifies (H.K.). In particular, a CM abelian variety A has supersingular reduction at all primes above p by [Sug12]. At every prime $\mathfrak{q} \mid p$ of F we construct signed Coleman maps and use their kernels to modify the local condition of the Selmer group at each of these primes. We remark that when A is not supersingular at all primes above p, we could still define signed Selmer groups by modifying the local conditions only at the primes where A has supersingular reduction. See Remark 5.8 for details.

1.2. **Notation and Hypotheses.** For any field k, let $\overline{k}$ denote a fixed separable closure of k and let $G_k = \mathrm{Gal}(\overline{k}/k)$ denote its absolute Galois group. For any positive integer n, let $\boldsymbol{\mu}_n$ denote the nth roots of unity and $\boldsymbol{\mu}_{p^\infty} = \varinjlim \boldsymbol{\mu}_{p^m}$.

Let F be a CM field and let K be its maximal real subfield as in the beginning of the introduction. For a general Dirichlet character $\chi : \mathrm{Gal}(\overline{F}/F) \to \mathfrak{O}^\times$, let $L = L_\chi$ denote the extension of F cut by χ. In this level of generality, we shall assume that

$$\text{the order of } \chi \text{ is prime to } p, \tag{1.1}$$

and

$$\chi(\wp) \neq 1 \quad \text{for any prime } \wp \text{ of } F \text{ above } p, \tag{1.2}$$

and that

$$\chi \neq \omega, \tag{1.3}$$

where ω is the Teichmüller character giving the action of G_F on $\boldsymbol{\mu}_p$. We will verify below that the character $\rho = \omega_\psi$ verifies these hypotheses.

Let $\mathcal{R}$ be the set of primes of F that does not contain any prime above p nor any prime at which χ is ramified. Define $\mathcal{N}(\mathcal{R})$ to be the square free products of primes chosen from $\mathcal{R}$. For $\ell \in \mathcal{R}$, let $F(\ell)$ be the maximal p-extension inside the ray class field of F modulo ℓ and for $\eta = \ell_1 \cdots \ell_s \in \mathcal{N}(\mathcal{R})$, set $F(\eta) = F(\ell_1) \cdots F(\ell_s)$. We write

[4]As we have also remarked in a previous footnote, Rubin-Stark elements suffice to deduce Theorem C.

$L(\eta) = L \cdot F(\eta)$ for the composite field. We define the collections of finite abelian extensions of F (resp., of L)

$$\mathfrak{E} = \{M \cdot F(\eta) : \eta \in \mathcal{N}(\mathcal{R}); M \subset F_\infty \text{ is a finite extension of } F\},$$

$$\mathfrak{E}_0 = \{M \cdot L(\eta) : \eta \in \mathcal{N}(\mathcal{R}); M \subset F_\infty \text{ is a finite extension of } F\},$$

Let $\mathfrak{K}_0 = \varinjlim_{N \in \mathfrak{E}_0} N$ and $\mathfrak{K} = \varinjlim_{N \in \mathfrak{E}} N$ and set $\mathfrak{G}(\mathfrak{X}) = \mathrm{Gal}(\mathfrak{X}/F)$.

For any non-archimedean prime λ of F, fix a decomposition group $\mathcal{D}_\lambda$ and the inertia subgroup $\mathcal{I}_\lambda \subset \mathcal{D}_\lambda$. Let $(-)^\vee = \mathrm{Hom}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ denote Pontryagin duality functor. Observe that $(-)^\vee \otimes \mathfrak{O} = \mathrm{Hom}(-, \Phi/\mathfrak{O})$. Bearing this relation in mind, we will write $X^\vee$ for $\mathrm{Hom}(X, \Phi/\mathfrak{O})$ *when X is an $\mathfrak{O}$-module.*

Let F_∞ and F^{cyc} be as above. Let F_n^{cyc} denote the unique subextension of F^{cyc}/F which has degree p^n and set $\Gamma_n = \mathrm{Gal}(F_n/F)$.

We let G_F act on Λ (resp., Λ_{cyc}) via the tautological surjection $G_F \twoheadrightarrow \Gamma$ (resp., $G_F \twoheadrightarrow \Gamma_{\mathrm{cyc}}$). For an $\mathfrak{O}$-module X of finite type which is endowed with a continuous action of G_F, we endow the Λ-module $X \otimes_{\mathfrak{O}} \Lambda$ by the diagonal G_F-action.

2. CM ABELIAN VARIETIES AND HECKE CHARACTERS

In this subsection we provide an overview of well-known facts about CM abelian varieties that we shall need below. They are originally due to Serre-Tate and Shimura. Let $A_{/F}$ be a principally polarized abelian variety which has CM by F. We assume that $\mathrm{End}_F(A)$ is an order in F whose index inside the maximal order is coprime to p. Suppose also that the field F contains the reflex field of A.

Let $T_p(A) = \varprojlim A[p^n]$ be the p-adic Tate-module of A. It is a free $\mathbb{Z}_p$-module of rank $2g$ on which G_F acts continuously. As explained in the Remark on page 502 of [ST68], $T_p(A)$ is free of rank one over $\mathcal{O}_F \otimes \mathbb{Z}_p = \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}$, where the product is over the primes of F that lie above p. This yields a decomposition $T_p(A) = \bigoplus_{\mathfrak{p}} T_{\mathfrak{p}}(A)$, where each $T_{\mathfrak{p}}(A) = \varprojlim A[\mathfrak{p}^n]$ is a free $\mathcal{O}_{\mathfrak{p}}$-module of rank one (and a $\mathbb{Z}_p$-module of rank $f(\mathfrak{p}/p)$, the inertia degree of $\mathfrak{p}$ over p). The G_F-action on $T_{\mathfrak{p}}(A)$ gives rise to a character

$$\psi_{\mathfrak{p}} : G_F \longrightarrow \mathcal{O}_{\mathfrak{p}}^\times.$$

By [Rib76, §2], $\psi_{\mathfrak{p}}$ is surjective for p large enough; we fix until the end a prime p satisfying this condition. We thence obtain a decomposition

$$T_p(A) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p = \bigoplus_{\mathfrak{p}|p} \bigoplus_{\sigma : F_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}}_p} V_{\mathfrak{p}}^\sigma,$$

where $V_{\mathfrak{p}}^\sigma$ is the one-dimensional $\overline{\mathbb{Q}}_p$-vector space on which G_F acts via the character $\psi_{\mathfrak{p}}^\sigma$, which is the compositum

$$G_F \xrightarrow{\psi_{\mathfrak{p}}} \mathcal{O}_{\mathfrak{p}}^\times \xrightarrow{\sigma} \overline{\mathbb{Q}}_p.$$

Fix an embedding $j_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $j_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let $\mathfrak{J} = \Sigma \cup \Sigma^c$ be the set of all embeddings of F into $\overline{\mathbb{Q}}$. Attached to A, there is a character

$$\boldsymbol{\psi} : \mathbb{A}_F/F^\times \longrightarrow F^\times,$$

which induces the Grössencharacters

$$\psi_\tau : \mathbb{A}_F/F^\times \xrightarrow{\boldsymbol{\psi}} F^\times \xrightarrow{j_\infty \circ \tau} \mathbb{C}^\times$$

and its p-adic avatars

$$\psi_\tau^{(p)} : \mathbb{A}_F/F^\times \xrightarrow{\boldsymbol{\psi}} F^\times \xrightarrow{j_p\circ\tau} \mathbb{C}_p^\times.$$

The theory of complex multiplication identifies the two sets $\{\mathrm{rec}\circ\psi_\tau^{(p)}\}_{\tau\in\mathfrak{J}}$ and $\{\psi_\mathfrak{p}^\sigma\}_{\mathfrak{p},\sigma}$ of p-adic Hecke characters, where $\mathrm{rec} : \mathbb{A}_F/F^\times \to G_F$ is the reciprocity map. Since we assume that the field F contains the reflex field of (F,Σ), the Hasse-Weil L-function $L(A/F,s)$ of A then factors into a product of Hecke L-series

$$L(A/F,s) = \prod_{\tau\in\mathfrak{J}} L(\psi_\tau, s).$$

Fix $\varepsilon \in \Sigma$ and identify F with F^ε. This choice in turn fixes a prime $\wp \in \Sigma_p$ and $\sigma : F_\wp \hookrightarrow \overline{\mathbb{Q}}_p$ in a way that $\mathrm{rec}\circ\psi_\varepsilon^{(p)} = \psi_\wp^\sigma$. Set $\mathfrak{O} := \sigma(\mathcal{O}_{F_\wp})$, let $\mathfrak{p} = \wp^\sigma$ denote its unique maximal ideal, $\mathfrak{F} := \mathrm{Frac}(\mathfrak{O})$ its fraction field and ϖ a fixed uniformizer. Define

$$\psi := \psi_\wp^\sigma : G_F \twoheadrightarrow \mathfrak{O}^\times. \tag{2.1}$$

Definition 2.1. Let $T = \mathfrak{O}(\psi) = T_\mathfrak{p}(A)$ denote the free $\mathfrak{O}$-module of rank one on which G_F acts via ψ. Let ω_ψ denote the character obtained as the compositum of the maps

$$G_F \xrightarrow{\psi} \mathfrak{O}^\times \longrightarrow (\mathfrak{O}/\mathfrak{p})^\times \xrightarrow{\tau} \mathfrak{O}^\times$$

where τ is the Teichmüller lift.

To ease notation we will sometimes write ρ in place of ω_ψ.

As we explain below in Remark 5.14, the character $\rho = \omega_\psi$ verifies the hypotheses (1.1), (1.2) and (1.3) for all sufficiently large primes p. For the main applications of this article towards the (signed) main conjectures for the CM abelian variety A, it will be sufficient to treat the main conjectures for the Dirichlet character $\chi = \rho$. We expect that with more work our approach may be generalized to treat the main conjectures for all Dirichlet characters χ verifying (1.1), (1.2) and (1.3).

Remark 2.2. For an element $f \in \mathrm{End}(A)\otimes_\mathbb{Z}\mathbb{Q} = F$, let f^ι denote its Rosati involution. Then, $f^\iota = \bar{f}$ is the endomorphism given by the complex conjugate of f viewed as an element of the CM field F. Furthermore, f^ι is the adjoint of f under the Weil pairing on $A[p^n]$ for every n. This fact in turn shows that the dual of the submodule $A[\mathfrak{p}^n]$ (under the restriction of the Weil pairing) is the submodule $A[\bar{\mathfrak{p}}^n]$. Thus T is self-dual (in the sense that $T \cong \mathrm{Hom}(T,\mathbb{Z}_p(1))$ as G_F-modules) if the prime $\mathfrak{p}_+ := \mathfrak{p}\cap K$ below $\mathfrak{p}$ is inert in F/K.

3. Semi-local preparation

Let $M = M_0\cdot F(\eta)$ be a member of the collection $\mathfrak{C}$, where M_0 is a finite subextension of F_∞/F. Set $\Delta_M = \mathrm{Gal}(M/F)$, $\delta_M = |\Delta_M|$ and $\Lambda_M = \mathfrak{O}[\Delta_M]$.

Let X be any $\mathfrak{O}[[G_F]]$-module which is free of rank d as an $\mathfrak{O}$-module. Suppose in addition that X satisfies the following hypothesis:

(H.p1) $H^2(F_\wp, X) = 0 = H^2(F_\wp, \mathrm{Hom}_\mathfrak{O}(X, \mathfrak{O}(1)))$, for any prime $\wp$ of F above p.

Lemma 3.1. *Suppose X is as above. Let $M \in \mathfrak{E}$ be an extension of F and let $\mathfrak{P}$ be a prime of M lying above p. Then*

$$H^2(M_{\mathfrak{P}}, X) = 0 = H^2\left(M_{\mathfrak{P}}, \mathrm{Hom}_{\mathfrak{O}}(X, \mathfrak{O}(1))\right).$$

Proof. Let $\wp$ be the prime of F lying below $\mathfrak{P}$ and set $D_{\mathfrak{P}} = \mathrm{Gal}(M_{\mathfrak{P}}/F_{\wp})$. Then either $D_{\mathfrak{P}}$ is trivial and in this case Lemma follows from **(H.p1)**, or otherwise $D_{\mathfrak{P}}$ is a non-trivial p-group. Then,

$$\#H^0(M_{\mathfrak{P}}, X^*[\varpi]) = \#H^0\left(D_{\mathfrak{P}}, (H^0(M_{\mathfrak{P}}, X^*[\varpi])\right) \equiv \#H^0(F_{\wp}, X^*[\varpi]) \equiv 1 \mod p$$

where the last equality holds thanks to **(H.p1)** and local duality. This shows that $H^0(M_{\mathfrak{P}}, X^*) = 0$ and thus by local duality that $H^2(M_{\mathfrak{P}}, X) = 0$, as desired. The second assertion is proved in an identical manner. □

Definition 3.2. For $j = 0, 1, 2$ define the semi-local cohomology groups

$$H^j(M_p, X) := \bigoplus_{\mathfrak{q}|p} H^j(M_{\mathfrak{q}}, X), \quad H^1_f(M_p, X) := \bigoplus_{\mathfrak{q}|p} H^1_f(M_{\mathfrak{q}}, X),$$

and let

$$\mathrm{loc}_p : H^1(M, X) \longrightarrow H^1(M_p, X)$$

denote the localization map.

Proposition 3.3. *Suppose* **(H.p1)** *holds true.*

(i) *The corestriction map*

$$\mathrm{cor} : H^1(M_p, X) \longrightarrow H^1(F_p, X)$$

is surjective.

(ii) *the Λ_M-module $H^1(M_p, X)$ is free of rank $2g \cdot d$.*

(iii) *The Λ-module $H^1(F_p, X \otimes \Lambda)$ is free of rank $2g \cdot d$.*

(iv) *The $\mathfrak{O}[[\mathfrak{G}(\mathfrak{K})]]$-module $\varprojlim_{M \in \mathfrak{E}} H^1(M_p, X)$ is free of rank $2g \cdot d$, where the inverse limits are with respect to corestriction maps.*

Proof. (iii) and (iv) follow at once from (i) and (ii). Both (i) and (ii) are essentially proved in [Büy13b, §2.1]. □

Let χ be a general Dirichlet character satisfying (1.1), (1.2) and (1.3) as before. Then the hypothesis **(H.p1)** holds true for $X = T_\chi$. In particular, the conclusions of Proposition 3.3 hold true for $X = T_\chi$. Set $\mathbb{T}_\chi := T_\chi \otimes \Lambda$.

Recall that $T = \mathfrak{O}(\psi) = T_{\mathfrak{p}}(A)$ and let $\mathbb{T} := T \otimes \Lambda$ and $\mathbb{T}_{\mathrm{cyc}} := T \otimes \Lambda_{\mathrm{cyc}}$. Then we have $T \cong T_\rho \otimes \langle \psi^{-1} \rangle$ (where $\rho = \omega_\psi$ is the Dirichlet character given as in Definition 2.1). We have the following twisting isomorphisms (c.f. [Rub00, Chapter VI])

$$H^1(X, \mathbb{T}_\rho) \otimes \langle \psi^{-1} \rangle \xrightarrow{\sim} H^1(X, \mathfrak{O}(1) \otimes \psi^{-1} \otimes \Lambda) \tag{3.1}$$

for $X = F$ or F_p.

Remark 3.4. As explained in Remark 2.2, the G_F-representation T is self-dual under our running assumptions and therefore

$$\mathfrak{O}(1) \otimes \psi^{-1} \cong \mathrm{Hom}(T, \mathfrak{O}(1)) \cong T$$

and the twisting isomorphisms (3.1) yields

$$H^1(X, \mathbb{T}_\rho) \otimes \langle \psi^{-1} \rangle \xrightarrow{\sim} H^1(X, \mathbb{T}) \tag{3.2}$$

for $X = F$ or F_p. By slight abuse, we denote any of the twisting isomorphisms in (3.2) by tw.

4. Rubin-Stark element Euler system of rank g

In this section, we review Rubin's [Rub96] integral refinement of Stark's conjectures. For the rest of this paper, we assume the truth of the Rubin-Stark conjecture [Rub96, Conjecture B′], whose content we explain below.

Let χ be a general Dirichlet character satisfying (1.1), (1.2) and (1.3) and let L denote the field cut by χ. Recall the definitions of the collections of extensions $\mathfrak{E}_0$ and $\mathfrak{E}$ from Section 1.2. Fix forever a finite set S of places of F that does *not* contain any prime above p, but contains the set of infinite places S_∞ and all primes $\lambda \nmid p$ at which χ is ramified. Assume that $|S| \geq g+1$. For each $\mathcal{K} \in \mathfrak{E}$, let

$$S_{\mathcal{K}} = \{\text{places of } \mathcal{K} \text{ that lie above } S\} \cup \{\text{places of } \mathcal{K} \text{ at which } \mathcal{K}/F \text{ is ramified}\}$$

be a set of places of $\mathcal{K}$. Let $\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}}}$ denote the $S_{\mathcal{K}}$ units of $\mathcal{K}$. Set $\Delta_{\mathcal{K}} := \mathrm{Gal}(\mathcal{K}/F)$ and $\delta_{\mathcal{K}} = |\mathrm{Gal}(\mathcal{K}/F)|$.

Definition 4.1. Let G be any finite group and let X be any $\mathfrak{O}[G]$-module which is of finite type over $\mathfrak{O}$. Following [Rub96], we define for any integer $r \geq 0$ the submodule $\wedge_0^r X \subset \Phi \otimes \wedge^r X$ by setting

$$\wedge_0^r X = \{x \in \Phi \otimes \wedge^r X : (\varphi_1 \wedge \cdots \wedge \varphi_r)(x) \in \mathfrak{O}[G] \quad \text{for every } \varphi_1, \cdots, \varphi_r \in \mathrm{Hom}(X, \mathfrak{O}[G])\}.$$

We also let $\overline{\wedge^r X}$ denote the isomorphic image of $\wedge^r X$ under the map $j : \wedge^r X \to \Phi \otimes \wedge^r X$.

Example 4.2. If X is a free $\mathfrak{O}[G]$-module then $\wedge_0^r X = \overline{\wedge^r X}$,

Conjecture B′ of [Rub96] predicts the existence of certain elements

$$\tilde{\varepsilon}_{\mathcal{K},S_{\mathcal{K}}} \in \Lambda_{\mathcal{K},S_{\mathcal{K}}} \subset \frac{1}{\delta_{\mathcal{K}}} \wedge^g \mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}}}$$

where the module $\Lambda_{\mathcal{K},S_{\mathcal{K}}}$ is defined in [Rub96, §2.1] and has the property that for any homomorphism

$$\tilde{\theta} \in \mathrm{Hom}_{\mathbb{Q}_p[\Delta_{\mathcal{K}}]}(\wedge^g \widehat{\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}}}} \otimes \mathbb{Q}_p, \widehat{\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}}}} \otimes \mathbb{Q}_p)$$

which is induced from a homomorphism

$$\theta \in \mathrm{Hom}_{\mathbb{Z}_p[\Delta_{\mathcal{K}}]}(\wedge^g \widehat{\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}}}}, \widehat{\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}}}}),$$

one has $\tilde{\theta}(\Lambda_{K,S_K}) \subset \widehat{\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}}}}$. We remark that the g-th exterior power $\wedge^g \widehat{\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}}}}$ (and all other exterior powers which appear below) is taken in the category of $\mathbb{Z}_p[\Delta_{\mathcal{K}}]$-modules.

Remark 4.3. Rubin's conjecture predicts that the elements $\tilde{\varepsilon}_{\mathcal{K},S_{\mathcal{K}}}$ should in fact lie inside the module $\frac{1}{\delta_{\mathcal{K}}} \wedge^g \mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}},\mathcal{T}}$, where $\mathcal{T} = \mathcal{T}_K$ is a finite set of primes disjoint from $S_{\mathcal{K}}$, chosen in a way that the group $\mathcal{O}^\times_{\mathcal{K},S_{\mathcal{K}},\mathcal{T}}$ of $S_{\mathcal{K}}$-units which are congruent to 1 modulo all the primes in $\mathcal{T}$ is torsion-free. One can safely ignore $\mathcal{T}$'s (as far as we are concerned in this paper) by simply setting $\mathcal{T}_L := \{\mathfrak{q}_1, \mathfrak{q}_2\}$ to consist of two primes of L which verify:

- The residue characteristics of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are distinct.
- $(\mathbf{N}\mathfrak{q}_1 - 1, p) = (\mathbf{N}\mathfrak{q}_2 - 1, p) = 1$.

One may then use [Rub96, §1.1] to show that the p-part of the leading coefficient of the modified zeta-function (at $s = 0$) is unaffected and also that $\widehat{\mathcal{O}_{L,S_L,\mathcal{T}_L}^\times} = \widehat{\mathcal{O}_{L,S_L}^\times}$.

We note that in [Büy09] one could choose the set $\mathcal{T}_L$ to be a singleton since one only deals with totally real fields in loc.cit.

Recall that F^{cyc} denotes the cyclotomic $\mathbb{Z}_p$-extension of F and for $m \in \mathbb{Z}^+$, F_m^{cyc} is the unique subextension of F of degree p^m.

Definition 4.4. For $\mathcal{K} = M \cdot L(\eta) \in \mathfrak{C}_0$ (or $\mathcal{K} = M \cdot F(\eta) \in \mathfrak{C}$), where $\eta \in \mathcal{N}(\mathcal{R})$ and $M \subset F_\infty$ a finite extension of F, choose $m \in \mathbb{Z}^+$ so that $M \not\subset F_m^{\text{cyc}}$ and set $M_m = M \cdot F_m^{\text{cyc}}$, $\mathcal{K}_m = \mathcal{K} \cdot M_m$. Define

$$\varepsilon_{\mathcal{K},S_\mathcal{K}} = \mathbf{N}^{\otimes g}_{\mathcal{K}_m/\mathcal{K}}\left(\tilde{\varepsilon}_{\mathcal{K}_m,S_{\mathcal{K}_m}}\right)$$

where $\mathbf{N}^{\otimes g}_{\mathcal{K}_m/\mathcal{K}}$ denotes the norm map induced on the g-th exterior power. It follows from [Rub96, Proposition 6.1] that $\varepsilon_{\mathcal{K},S_\mathcal{K}}$ is well-defined.

As we have fixed S (therefore $S_\mathcal{K}$ as well), we will often drop S or $S_\mathcal{K}$ from the notation and denote $\varepsilon_{\mathcal{K},S_\mathcal{K}}$ by $\varepsilon_\mathcal{K}$; or sometimes use S instead of $S_\mathcal{K}$ and denote $\mathcal{O}_{\mathcal{K},S_\mathcal{K}}$ by $\mathcal{O}_{\mathcal{K},S}$.

For any number field $\mathcal{K}$, Kummer theory gives a canonical isomorphism

$$H^1(\mathcal{K}, \mathfrak{O}(1)) \cong \widehat{\mathcal{K}^\times} \otimes_{\mathbb{Z}_p} \mathfrak{O} := \left(\varprojlim_n \mathcal{K}^\times/(\mathcal{K}^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} \mathfrak{O}.$$

Under this identification, we view each $\varepsilon_{\mathcal{K},S_\mathcal{K}}$ as an element of $\frac{1}{\delta_\mathcal{K}} \wedge^g H^1(\mathcal{K}, \mathfrak{O}(1))$. The distribution relation satisfied by the Rubin-Stark elements ([Rub96, Proposition 6.1]) shows that the collection $\{\varepsilon_{\mathcal{K},S_\mathcal{K}}\}_{K\in\mathcal{K}}$ is an Euler system of rank g in the sense of [PR98], as appropriately generalized in [Büy10, Definition 3.1] to allow denominators.

4.1. **Twisting.** Let $T_\chi := \mathfrak{O}(1) \otimes \chi^{-1}$. We may *twist* the collection $\{\varepsilon_{\mathcal{K},S_\mathcal{K}}\}_{\mathcal{K}\in\mathfrak{C}}$ we have obtained above (which is a rank-g Euler system for the collection $\mathfrak{O}(1)$), following the formalism of [Rub00, §II.4] (as done so in [Büy13b, §3.1]). We do not include the details here and note only the following identification (for every $\mathcal{K} \in \mathfrak{C}$)

$$H^1(\mathcal{K}, T_\chi) \xrightarrow{\sim} L\mathcal{K}^{\times,\chi}$$

obtained using the inflation-restriction sequence and Kummer theory.

Let $\varepsilon_\mathcal{K}^\chi \in \wedge^g H^1(\mathcal{K}, T_\chi)$ denote the twisted element. Then the collection $\mathcal{C}^{(g)}_{\text{R-S}} := \{\varepsilon_\mathcal{K}^\chi\}_{\mathcal{K}\in\mathfrak{C}}$ is a rank-g Euler system for T_χ (in the sense of [Büy10, Definition 3.1]) and we call it the *Rubin-Stark element Euler system of rank* g.

Remark 4.5. Let $\mathcal{K}$ be any field contained in the collection $\mathfrak{C}$. When χ verifies (1.1), Proposition 3.3 and Example 4.2 shows that

$$\text{loc}_p(\varepsilon_\mathcal{K}^\chi) \in \overline{\wedge^g H^1(\mathcal{K}_p, T_\chi)}$$

where the exterior product is taken in the category of $\mathfrak{O}[\mathrm{Gal}(\mathcal{K}/F)]$-modules. We will simply write $\mathrm{loc}_p(\varepsilon_{\mathcal{K}}^{\chi})$ in place of $j^{-1}(\mathrm{loc}_p(\varepsilon_{\mathcal{K}}^{\chi})) \in \wedge^g H^1(\mathcal{K}_p, T_\chi)$.

Definition 4.6. We define

$$\mathrm{loc}_p(\varepsilon_{F_\infty}^{\chi}) = \{\mathrm{loc}_p(\varepsilon_M^{\chi})\} \in \varprojlim \wedge^g H^1(M_p, T_\chi) = \wedge^g \varprojlim H^1(M_p, T_\chi) = \wedge^g H^1(F_p, \mathbb{T}_\chi)$$

to be the tower of Rubin-Stark elements along F_∞. Here the inverse limit is taken over all finite subextensions of F_∞/F and the second equality holds thanks to the fact that each module $H^1(M_p, T_\chi)$ is free as an $\mathfrak{O}[\mathrm{Gal}(M/F)]$-module.

Definition 4.7. In what follows, we denote a generic $\mathbb{Z}_p$-power extension of F that is disjoint from F^{cyc} by F_o. We further write $\Gamma_o = \mathrm{Gal}(F_o/F)$. We fix a basis (as a $\mathbb{Z}_p$-module) $\{\gamma_1, \cdots, \gamma_s\}$ of Γ_o (where $s \le g + \mathfrak{d}$ is a non-negative integer) and let γ_{cyc} denote a fixed topological generator of Γ^{cyc}. We set $\Lambda_o = \mathfrak{O}[[\Gamma_o]]$.

Given a positive integer m and an s-tuple of positive integers $\overline{n} = (n_1, \cdots, n_s)$ we let $F_{\mathrm{cyc}} \subset F_{\overline{n}} \subset F_\infty$ denote the fixed field of $\Gamma_o^{p^{\overline{n}}} := \langle \gamma_1^{p^{n_1}} \rangle \times \cdots \times \langle \gamma_s^{p^{n_s}} \rangle$ and let $F \subset F_{m,\overline{n}} \subset F_{\overline{n}}$ be the fixed field of $\Gamma_{\mathrm{cyc}}^{p^m}$. Set $\Gamma_{(m)} = \Gamma_{\mathrm{cyc}}/\Gamma_{\mathrm{cyc}}^{p^m}$ and $\Gamma^{(\overline{n})} = \Gamma_o/\Gamma_o^{p^{\overline{n}}}$. We write $F_{(m)} = F_{m,\overline{0}} \subset F^{\mathrm{cyc}}$ (where $\overline{0} = (0, \cdots, 0)$) and $F^{(\overline{n})} = F_{0,\overline{n}} \subset F_o$. Observe that $F_{m,\overline{n}}$ is the joint of $F_{(m)}$ and $F^{(\overline{n})}$.

4.2. **Perrin-Riou-Stark Conjecture and an explicit reciprocity conjecture.** We shall use the notation we have set in Definition 4.7 throughout this subsection. Let χ be a Dirichlet character satisfying (1.1), (1.2) and (1.3).

Definition 4.8. The *canonical Selmer structure* $\mathcal{F}_{\mathrm{can}}$ is given by the choice of local conditions $H^1_{\mathcal{F}_{\mathrm{can}}}(F_\mathfrak{q}, \mathbb{T}_\chi) = H^1(F_\mathfrak{q}, \mathbb{T}_\chi)$, for all primes $\mathfrak{q}$ of F. For every finite sub-extension $F \subset M \subset F_\infty$, we define the the propagation of $\mathcal{F}_{\mathrm{can}}$ to the G_M-representation T_χ by setting (for every prime $\mathfrak{q}$ of F)

$$H^1_{\mathcal{F}_{\mathrm{can}}}(M_\mathfrak{q}, T_\chi) := \mathrm{im}\left(H^1(F_\mathfrak{q}, \mathbb{T}_\chi) = \varprojlim_{F \subset N \subset F_\infty} H^1(N_\mathfrak{q}, T_\chi) \longrightarrow H^1(M_\mathfrak{q}, T_\chi) \right).$$

It is not hard to see (for the first we use Proposition 3.3(i) and for the second, Corollary B.3.4 and Lemma 1.3.5(iii) of [Rub00]) that:

- For every $\mathfrak{q} \mid p$ we have $H^1_{\mathcal{F}_{\mathrm{can}}}(M_\mathfrak{q}, T_\chi) = H^1(M_\mathfrak{q}, T_\chi)$.
- For every $\mathfrak{q} \nmid p$ we have $H^1_{\mathcal{F}_{\mathrm{can}}}(M_\mathfrak{q}, T_\chi) = H^1_f(M_\mathfrak{q}, T_\chi)$, where

$$H^1_f(M_\mathfrak{q}, T_\chi) := \ker\left(H^1(M_\mathfrak{q}, T_\chi) \longrightarrow H^1(M_\mathfrak{q}^{\mathrm{ur}}, T_\chi \otimes \mathbb{Q}_p)\right)$$

  and $M_\mathfrak{q}^{\mathrm{ur}}$ is the maximal unramified extension of $M_\mathfrak{q}$.

Example 4.9. For a general Dirichlet character χ of F, we may identify $H^1(F, T_\chi)$ with $L^{\times,\chi}$ and similarly, for any rational prime ℓ, the semi-local cohomology group $H^1(F_\ell, T_\chi)$ with $(L \otimes \mathbb{Q}_\ell)^{\times,\chi}$ by Kummer theory.

Set $\mathcal{U}_\ell = (\mathcal{O}_L \otimes \mathbb{Z}_\ell)^{\times,\chi}$. It follows from [Rub00, §1.6.C and Prop. B.3.3] along with the proof of Prop. 3.2.6 of loc.cit. that $H^1_{\mathcal{F}_{\mathrm{can}}}(F_\ell, T_\chi) = \mathcal{U}_\ell$, for every rational prime ℓ. Note that this holds true even for $\ell = p$, thanks to the our running assumption (1.2). We therefore conclude that $H^1_{\mathcal{F}_{\mathrm{can}}}(F, T_\chi) = \mathcal{O}_L^{\times,\chi}$. It follows from [NSW08, §8.6.12] that the $\mathfrak{O}$-module $\mathcal{O}_L^{\times,\chi}$ is free of rank g, since χ is not the Teichmüller character.

Similarly, $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,T^*_\chi)^\vee \cong \mathrm{Cl}(L)^\chi$.

The following is Lemme 4 of Section 1.3 in [PR84]:

Lemma 4.10 (Perrin-Riou). *Let L_2/F be an extension such that $G := \mathrm{Gal}(L_2/F) \cong \mathbb{Z}_p^{s+1}$ and L_1/F a subextension of L_2 with $H := \mathrm{Gal}(L_2/L_1) \cong \mathbb{Z}_p$ and $G/H \cong \mathbb{Z}_p^s$. For $X = G$ or G/H, let $\Lambda(X)$ stand for the Iwasawa algebra $\mathfrak{O}[[X]]$. Let $\pi_H : \Lambda(G) \to \Lambda(G/H)$ denote the natural projection and let γ_H be a fixed topological generator of H. Suppose that M is a torsion $\Lambda(G)$-module. Set $M_H := M \otimes \Lambda(G/H) \cong M/(\gamma_H - 1)M$ and $M^H := M[\gamma_H - 1]$.*

- (i) *M_H is $\Lambda(G/H)$-torsion iff $\pi_H(\mathrm{char}_{\Lambda(G)}(M)) \neq 0$ iff $(\gamma_H - 1)$ is prime to $\mathrm{char}_{\Lambda(G)}(M)$.*
- (ii) *If M_H is $\Lambda(G/H)$-torsion, then M^H is a pseudo-null $\Lambda(G)$-module and a torsion $\Lambda(G/H)$-module. In this case*

$$\pi_H(\mathrm{char}_{\Lambda(G)}(M)) \cdot \mathrm{char}_{\Lambda(G/H)}(M^H) = \mathrm{char}_{\Lambda(G/H)}(M_H)\,.$$

Lemma 4.11. *Suppose that the weak Leopoldt conjecture holds true for L. Then the Λ_{cyc}-module $H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi \otimes \Lambda_{\mathrm{cyc}})$ and the Λ-module $H^1_{\mathcal{F}_{\mathrm{can}}}(F,\mathbb{T}_\chi)$ are both free of rank g.*

Proof. The fact that $H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi \otimes \Lambda_{\mathrm{cyc}})$ is a Λ_{cyc}-module of rank g is a direct consequence of the weak Leopoldt conjecture for L. Let γ_{cyc} be a topological generator of Γ^{cyc}. To see that the module $H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi \otimes \Lambda_{\mathrm{cyc}})$ is in fact free, observe that the augmentation map induces an injective map

$$H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi \otimes \Lambda_{\mathrm{cyc}})/(\gamma_{\mathrm{cyc}} - 1) \hookrightarrow H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi)$$

by the discussion in §.1.6.C, Proposition B.3.3 along with the proof of Proposition 3.2.6 of [Rub00]. Note that in order to compare local conditions at p, we rely on our assumption (1.2). This and an easy extension of Dirichlet's unit theorem shows by Nakamaya's lemma that the Λ_{cyc}-module $H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi \otimes \Lambda_{\mathrm{cyc}})$ may be generated by at most g elements. Using the fact that this module is torsion free, it is not hard to see that these generators cannot satisfy a non-trivial Λ_{cyc}-linear relation. This completes the proof of the assertion regarding the module $H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi \otimes \Lambda_{\mathrm{cyc}})$.

Let now F_o/F be any $\mathbb{Z}_p$-power extension as in Definition 4.7. The notation we use here also follows our set up there. Let $F_\dagger = F^{\mathrm{cyc}}F_o$, $\Gamma_\dagger = \mathrm{Gal}(F_\dagger/F)$ and $\Lambda_\dagger = \mathfrak{O}[[\Gamma_\dagger]]$. Let $\widetilde{R\Gamma}_{f,\mathrm{Iw}}(F_\dagger/F,T)$ be Nekovář's Selmer complex associated to $T_\chi \otimes \Lambda_\dagger$, which is given by the Greenberg local conditions determined by the choice $U_v^+ = T_\chi$ for every prime v of F above p. As we have assumed (1.2), it follows from [Nek06, Lemma 9.6.3] (and [Nek06, Proposition 8.8.6] used in order to pass to limit) that

$$\widetilde{H}^1_{f,\mathrm{Iw}}(F_\dagger/F,T_\chi) \xrightarrow{\sim} H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi \otimes \Lambda_\dagger)$$

where $\widetilde{H}^1_{f,\mathrm{Iw}}$ denotes the cohomology of the Selmer complex in degree 1. Under the hypothesis (1.2), Nekovář proved that the Selmer complex may be represented by a perfect complex concentrated in degrees 1 and 2. In particular, its cohomology $H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi \otimes \Lambda_\dagger)$ in degree 1 is a projective (hence free) $\Lambda_\dagger$-module. By Nakayama's lemma, it may be generated by at most g elements. We will show inductively (on the Krull dimension of $\Lambda_\dagger$) below that it cannot admit a set of generators of size strictly smaller than g. This will conclude the proof of our lemma.

Let $F_o' \subset F_o$ be a sub-$\mathbb{Z}_p$-power extension of F that is disjoint from F^{cyc} and such that $\mathrm{Gal}(F_o/F_o') \cong \mathbb{Z}_p$. Let $\gamma_\dagger$ be a topological generator of $\mathrm{Gal}(F_o/F_o')$. Let $F_\dagger' = F_o'F^{\mathrm{cyc}}$.

Set $\Gamma'_\dagger = \mathrm{Gal}(F'_\dagger/F)$ and $\Lambda'_\dagger = \mathfrak{O}[[\Gamma'_\dagger]]$. Suppose that we have already proved that the $\Lambda'_\dagger$-module $\widetilde{H}^1_{f,\mathrm{Iw}}(F'_\dagger/F, T_\chi)$ is free of rank g. Then:

$$\text{(4.1)}\qquad \mathrm{coker}\left(\widetilde{H}^1_{f,\mathrm{Iw}}(F_\dagger/F, T_\chi) \longrightarrow \widetilde{H}^1_{f,\mathrm{Iw}}(F'_*/F, T_\chi)\right) \cong \widetilde{H}^2_{f,\mathrm{Iw}}(F_\dagger/F, T_\chi)[\gamma_\dagger - 1]$$
$$\cong H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T_\chi \otimes \Lambda_\dagger)^*)^\vee[\gamma_\dagger - 1]$$

Here the first isomorphism follows from Nekovář's control theorem [Nek06, 8.10.1]; second from his duality theorem [Nek06, 8.9.6.2]. One may identify the $\Lambda_\dagger$-module $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T_\chi \otimes \Lambda_\dagger)^*)^\vee$ with $\varprojlim_{L\subset M\subset LF_\dagger} \mathrm{Cl}(M)^\chi$ and argue using Lemma 4.10 that the cokernel module (4.1) is $\Lambda_\dagger$-torsion. Therefore, the $\Lambda_\dagger$-module $H^1_{\mathcal{F}_{\mathrm{can}}}(F, T \otimes \Lambda_\dagger)$ cannot be generated by less than g elements. □

Proposition 4.12. (i) $\mathrm{coker}\left(H^1(F, T_\chi \otimes \Lambda_{\mathrm{cyc}}) \to H^1_{\mathcal{F}_{\mathrm{can}}}(F_{(m)}, T_\chi)\right)$ *is finite.*
(ii) *The* $\mathfrak{O}[\Gamma_{(m)}]$*-module* $H^1_{\mathcal{F}_{\mathrm{can}}}(F_{(m)}, T_\chi)$ *is free of rank* g*.*
(iii) *The* $\mathfrak{O}[\Gamma_{(m)} \times \Gamma^{(\overline{n})}]$*-module* $H^1_{\mathcal{F}_{\mathrm{can}}}(F_{(m,\overline{n})}, T_\chi)$ *is free of rank* g*.*

Proof. We argue as in Lemma 4.11. By Nekovář's control theorem

$$\mathrm{coker}\left(H^1(F, T_\chi \otimes \Lambda_{\mathrm{cyc}}) \to H^1_{\mathcal{F}_{\mathrm{can}}}(F_{(m)}, T_\chi)\right) \cong \widetilde{H}^2_{f,\mathrm{Iw}}(F^{\mathrm{cyc}}/F, T_\chi)[\gamma_{\mathrm{cyc}}^{p^m} - 1]$$

and $\widetilde{H}^2_{f,\mathrm{Iw}}(F^{\mathrm{cyc}}/F, T_\chi) \cong H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T_\chi \otimes \Lambda_{\mathrm{cyc}})^*)^\vee$. Since

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T_\chi \otimes \Lambda_{\mathrm{cyc}})^*)^\vee/(\gamma_{\mathrm{cyc}}^{p^m} - 1) \cong H^1_{\mathcal{F}^*_{\mathrm{can}}}(F_{(m)}, T^*_\chi)^\vee \cong \mathrm{Cl}(LF_{(m)})^\chi$$

is finite, the characteristic ideal of the torsion Λ_{cyc}-module $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T_\chi \otimes \Lambda_{\mathrm{cyc}})^*)^\vee$ is prime to $\gamma_{\mathrm{cyc}}^{p^m} - 1$, and by the structure theorem for finitely generated Λ_{cyc}-modules we see that $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T_\chi \otimes \Lambda_{\mathrm{cyc}})^*)^\vee[\gamma_{\mathrm{cyc}}^{p^m} - 1]$ is finite, concluding the proof of (i).

The argument above also shows that $\mathrm{coker}(H^1(F, T_\chi \otimes \Lambda_{\mathrm{cyc}}) \to H^1_{\mathcal{F}_{\mathrm{can}}}(F, T_\chi))$ is finite, which in turn implies that $\mathrm{coker}\left(H^1(F_{(m)}, T_\chi) \xrightarrow{pr} H^1_{\mathcal{F}_{\mathrm{can}}}(F, T_\chi)\right)$ is finite as well. We therefore infer that the image of pr (induced by projection modulo $\gamma_{\mathrm{cyc}} - 1$) is a free $\mathfrak{O}$-module of rank g. It follows by Nakayama's lemma that the $\mathfrak{O}[\Gamma_{(m)}]$-module $H^1(F_{(m)}, T_\chi)$ may be generated by at most g elements, say by $\{v_1, \cdots, v_g\}$. On the other hand, it follows from the first part that $H^1(F_{(m)}, T_\chi)$ contains a free $\mathfrak{O}[\Gamma_{(m)}]$-module of rank g (isomorphic image of the free module $H^1(F, T_\chi \otimes \Lambda_{\mathrm{cyc}})/(\gamma_{\mathrm{cyc}}^{p^m} - 1)$), say with basis $\{y_1, \cdots, y_g\}$. One may easily verify that any non-trivial $\mathfrak{O}[\Gamma_{(m)}]$-linear relation $\{v_1, \cdots, v_g\}$ would yield a non-trivial $\mathfrak{O}[\Gamma_{(m)}]$-linear relation of $\{y_1, \cdots, y_g\}$, which is impossible. This shows that $\{v_1, \cdots, v_g\}$ is indeed a basis and (ii) follows.

The proof of (iii) follows by induction on the Krull dimension $s + 1$ of Λ_o (the base case being (ii)). We indicate the main steps:

- Suppose we have verified for the $(s-1)$-tuple $\overline{n}' = (n_1, \cdots, n_{s-1})$ and the non-negative integer m that the $\Lambda_{(m,\overline{n}')} := \mathfrak{O}[\Gamma_{(m)} \times \Gamma^{(\overline{n}')}]$-module $H^1_{\mathcal{F}_{\mathrm{can}}}(F_{(m,\overline{n}')}, T_\chi)$ is free of rank g. Let Γ_s be the subgroup of Γ_o topologically generated by γ_s and set $\Lambda_s = \mathfrak{O}[[\Gamma_s]]$. Using Nakayama's Lemma along with Nekovář's descent as above, prove that the $\Lambda_{(m,\overline{n}')}[[\Gamma_s]]$-module $H^1_{\mathcal{F}_{\mathrm{can}}}(F_{(m,\overline{n}')}, T_\chi \otimes \Lambda_s)$ is free of rank g as well.
- Use once again Nekovář's descent and finiteness of the ideal class groups to show that $\mathrm{coker}(H^1_{\mathcal{F}_{\mathrm{can}}}(F_{(m,\overline{n}')}, T_\chi \otimes \Lambda_s) \to H^1_{\mathcal{F}_{\mathrm{can}}}(F_{(m,\overline{n})}, T_\chi))$ is finite. Conclude

that $H^1_{\mathcal{F}_{\mathrm{can}}}(F_{(m,\overline{n})},T_\chi))$ contains a free $\mathfrak{O}[\Gamma_{(m)}\times\Gamma^{(\overline{n})}]$-module of rank g (with finite index in $H^1_{\mathcal{F}_{\mathrm{can}}}(F_{m,\overline{n}},T_\chi)$), say again with basis $\{y_1,\cdots,y_g\}$.
- Furthermore, it follows by Nakayama's lemma that $H^1_{\mathcal{F}_{\mathrm{can}}}(F_{m,\overline{n}},T_\chi)$ may be generated by at most g elements, say by $\{v_1,\cdots,v_g\}$. It can be verified that a non-trivial linear relation of $\{v_1,\cdots,v_g\}$ would yield a non-trivial relation among $\{y_1,\cdots,y_g\}$, concluding the proof that $\{v_1,\cdots,v_g\}$ is a basis of $H^1_{\mathcal{F}_{\mathrm{can}}}(F_{m,\overline{n}},T_\chi)$.

□

Henceforth we shall take s in Definition 4.7 to be $g+1$, so that $F_o\,F^{\mathrm{cyc}}=F_\infty$. In particular, note that all tuples $\overline{n}$ will consist of $g+1$ non-negative integers.

Remark 4.13. By Proposition 4.12(iii) it follows that

$$\varepsilon^\chi_{F_{m,\overline{n}}}\in\wedge^g H^1_{\mathcal{F}_{\mathrm{can}}}(F_{m,\overline{n}},T_\chi)\,,$$

since we have $\wedge^g_0 H^1_{\mathcal{F}_{\mathrm{can}}}(F_{m,\overline{n}},T_\chi)=\wedge^g H^1_{\mathcal{F}_{\mathrm{can}}}(F_{m,\overline{n}},T_\chi)$ by Example 4.2.

Inspired by [PR98, Definition 1.2.2], we propose the following strengthening (along the tower F_∞/F) of the Rubin-Stark conjectures:

Conjecture 4.14 (Perrin-Riou-Stark conjecture)**.** *There exists an element*

$$\mathfrak{S}^\chi_\infty=\mathfrak{S}_{\infty,1}\wedge\cdots\wedge\mathfrak{S}_{\infty,g}\in\wedge^g H^1(F,\mathbb{T}_\chi)$$

(where the exterior product is evaluated in the category of Λ-modules) such that for every subextension $F\subset M=F_{m,\overline{n}}\subset F_\infty$ as above, the image of $\mathfrak{S}_\infty$ under the natural projection to $\wedge^g H^1_{\mathcal{F}_{\mathrm{can}}}(M,T_\chi)$ is ε^χ_M, the χ-isotypic component of the Rubin-Stark element.

Assuming the truth of the Perrin-Riou-Stark conjecture, we set

$$\mathfrak{S}^\chi_{\mathrm{cyc}}=\mathfrak{S}_{\mathrm{cyc},1}\wedge\cdots\wedge\mathfrak{S}_{\mathrm{cyc},g}\in\wedge^g H^1(F,T_\chi\otimes\Lambda_{\mathrm{cyc}})$$

to denote the image of $\mathfrak{S}_\infty$.

Remark 4.15. If there is a chain of $\mathbb{Z}_p$-power extensions

$$F=\mathfrak{F}_0\subset\mathfrak{F}_1\subset\cdots\mathfrak{F}_{g+1}=F_\infty$$

for which we have $\mathrm{Gal}(\mathfrak{F}_j/\mathfrak{F}_{j-1})\cong\mathbb{Z}_p$, such that for any $j=1,\cdots g+1$ the module $\varprojlim_{L\subset M\subset L\mathfrak{F}_j}\mathrm{Cl}(M)^\chi$ has no $\mathfrak{O}[[\mathrm{Gal}(\mathrm{Gal}(\mathfrak{F}_j/F))]]$-pseudo-null submodules, then the Perrin-Riou-Stark conjecture would have been trivial. Indeed in that case, it follows that for non-negative integers $m\geq m'$ and g-tuples of non-negative integers $\overline{n}=(n_1,\cdots,n_g)$, $\overline{n}'=(n'_1,\cdots,n'_g)$ with $n_i\geq n'_i$ for every i, the maps

$$H^1_{\mathcal{F}_{\mathrm{can}}}(F_{m,\overline{n}},T_\chi)\longrightarrow H^1_{\mathcal{F}_{\mathrm{can}}}(F_{m',\overline{n}'},T_\chi)$$

(and $n_i\geq n'_i$, $1\leq i\leq g$) are surjective and using Proposition 4.12 we conclude that

$$\varprojlim\wedge^g H^1_{\mathcal{F}_{\mathrm{can}}}(F_{m,\overline{n}},T_\chi)=\wedge^g\varprojlim H^1_{\mathcal{F}_{\mathrm{can}}}(F_{m,\overline{n}},T_\chi)=\wedge^g H^1(F,\mathbb{T}_\chi)\,.$$

In our companion article [BL14, Conjecture 3.12] we have predicted a certain form of an explicit reciprocity conjecture (which in turn itself is based on Perrin-Riou's conjectures on p-adic L-functions) for a class of motives. We shall recall the explicit reciprocity conjecture of loc.cit. adapted to the setting of our current article, where the motive in question is that of an Grössencharacter associated to a CM abelian variety. In particular, we conjecture that the *Kolyvagin determinants* verifying Conjecture 3.10 of loc.cit. should come from Rubin-Stark elements via the recipe in Appendix B.

Definition 4.16. Let us now choose our Dirichlet character χ to be $\rho = \omega_\psi$. We assume the truth of the Perrin-Riou-Stark Conjecture 4.14. Let $\mathfrak{U}_\infty \subset \wedge^g H^1_{\mathcal{F}_{\text{can}}}(F, \mathbb{T}_\rho)$ denote the Λ-module generated by the Perrin-Riou-Stark element $\mathfrak{S}^\rho_\infty$. Let $\mathfrak{U}^\psi_\infty$ be the image of $\mathfrak{U}$ under the twisting isomorphism

$$\wedge^g H^1_{\mathcal{F}_{\text{can}}}(F, \mathbb{T}_\rho) \otimes \langle \psi^{-1} \rangle \xrightarrow{\sim} \wedge^g H^1_{\mathcal{F}_{\text{can}}}(F, \mathbb{T}). \tag{4.2}$$

The module $\mathfrak{U}^\psi_\infty$ is called the module of ψ-twisted Perrin-Riou-Stark elements.

For any unramified extension $K/\mathbb{Q}_p$ containing a completion of F (at a prime above p), let $\mathbb{D}_K(T)$ denote the Dieudonné module and

$$[\sim, \sim] : \mathbb{D}_K(T) \times \mathbb{D}_K(\mathrm{Hom}(T, \mathbb{Z}_p(1))) \longrightarrow \mathbb{Z}_p$$

denote the natural pairing. As explained in Appendix C, the Dieudonné module $\mathbb{D}_K(T)$ is endowed with a natural $\mathfrak{O}$-module structure, respecting the action of φ and the filtration. For $\mathfrak{q}|p$ a prime of F above p, let $\mathfrak{B}_\mathfrak{q} = \{v_{\mathfrak{q},i}\}_{i=1}^{f(\mathfrak{q}/\mathfrak{p})}$ be an $\mathfrak{O}$-basis of $\mathbb{D}_{F_\mathfrak{q}}(T)$. The dual basis of $\mathfrak{B}$ is then denoted by $\mathfrak{B}' = \left\{v'_{\mathfrak{q},i}\right\}_{i=1}^{f(\mathfrak{q}/p)}$.

Definition 4.17. Given $\mathbf{c} = \mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_g \in \wedge^g H^1(F, \mathbb{T}_{\text{cyc}})$, θ an even Dirichlet character of conductor p^n and for $I \in \mathfrak{I}$, we write $\mathfrak{M}^I_\theta(\mathbf{c})$ for the $g \times g$ matrix whose entries are given by $\left[\sum_{\sigma \in \Gamma_n} \theta(\sigma) \exp^*_n \left(\mathrm{loc}_\mathfrak{p}(\mathbf{c}_i)^\sigma\right), v'_{\mathfrak{p},j}\right]$ for $1 \leq i \leq g$ and $j \in I_\mathfrak{p}$.

We conjecture that a generator of the module of ψ-twisted Perrin-Riou-Stark elements verify the following explicit reciprocity conjecture of [BL14] (which in turn is based on a conjecture of Perrin-Riou):

Conjecture 4.18. *There exists a generator $\boldsymbol{\xi}$ of $\mathfrak{U}^\psi_\infty$ such that for every $n \in \mathbb{Z}^+$ and a primitive character $\theta : \Gamma_n \to \boldsymbol{\mu}_{p^\infty}$ we have*

$$\det\left(\mathfrak{M}^I_\theta(\boldsymbol{\xi}_{\text{cyc}})\right) = L_{\{p\}}(\psi, \theta^{-1}, 1) \cdot \frac{\Omega^I_{\psi\theta^{-1},p}}{\Omega^I_{\psi\theta^{-1}}}$$

where $\boldsymbol{\xi}_{\text{cyc}} = \xi_1 \wedge \cdots \wedge \xi_g \in \wedge^g H^1(F, \mathbb{T}_{\text{cyc}})$ is the image of $\boldsymbol{\xi}$, $L_{\{p\}}(\psi, \theta^{-1}, s)$ denotes the complex L-function $L_{\{p\}}(\psi, \theta^{-1}, s)$ with the Euler factor at p removed and $\Omega^I_{\psi\theta^{-1}}$ (resp., $\Omega^I_{\psi\theta^{-1},p}$) is Deligne's complex (resp., Perrin-Riou's p-adic) period (see [BL14, Definition 3.5]*). Furthermore, there exists a Dirichlet character θ with $L(\psi, \theta^{-1}, 1) \neq 0$.*

5. Integral Iwasawa theory for supersingular CM abelian varieties

In this section we apply the integral Iwasawa theory studied extensively in [BL14] to the representation $T = T_\mathfrak{p}(A)$. For notational simplicity, we shall write R in place of Λ_{cyc}.

Let $\mathfrak{q}$ (which may or may not coincide with $\mathfrak{p}$) be a prime of F that lies above p. We write $\mathbb{D}_\mathfrak{q}(T)$ for the Dieudonné module of the representation $T|_{G_{F_\mathfrak{q}}}$ and we denote $\oplus_{\mathfrak{q}|p}\mathbb{D}_\mathfrak{q}(T)$ by $\mathbb{D}_p(T)$.

Lemma 5.1. *The inertia degree $f(\mathfrak{p}/p)$ is even.*

Proof. Let $G_{\mathfrak{p}} = \varinjlim A[\mathfrak{p}^n]$. Since A is supersingular at p, this is isogenous to $G_{1,1}^{\oplus r_{\mathfrak{p}}}$ for some integer $r_{\mathfrak{p}}$, where $G_{1,1}$ denotes the p-divisible group of a supersingular elliptic curve, which has height 2. The height of $G_{\mathfrak{p}}$ is given by $f(\mathfrak{p}/p)$, which should equal to $2r_{\mathfrak{p}}$. Hence the result. □

Lemma 5.2. *The Hodge-Tate weights of $T|_{G_{F_{\mathfrak{q}}}}$ are 0 and 1. The eigenvalues of φ on $\mathbb{D}_{\mathfrak{q}}(T_{\mathfrak{p}}(A))$ are of the form $\zeta \times p^{-\frac{1}{2}}$ where ζ is a root of unity.*

Proof. Since the Hodge-Tate weights of $T_p(A)$ are 0 and 1 (with multiplicity g) and T is a subrepresentation of $T_p(A)$, the first part of the lemma follows. The second part of the lemma is a theorem of Manin-Oort. □

In particular, the representation $T|_{G_{F_{\mathfrak{q}}}}$ satisfies the hypotheses (H.F.-L.) and (H.S.) in [BL14, §2.1]. As explained in Appendix C, we obtain an R-equivariant Coleman map

$$\mathrm{Col}_{\mathfrak{p}}^{\mathfrak{q}} : H^1(F_{\mathfrak{q}}, \mathbb{T}_{\mathrm{cyc}}) \longrightarrow R^{[F_{\mathfrak{q}}:\mathbb{Q}_p]} .$$

Note that in [BL14], we have defined a Coleman map on $\varprojlim H^1(F_{\mathfrak{q}}(\boldsymbol{\mu}_{p^n}), T)$, so our Coleman map here is in fact the isotypic component of that in op. cit corresponding to the trivial character on $\mathrm{Gal}(F_{\mathfrak{q}}(\boldsymbol{\mu}_p)/F_{\mathfrak{q}})$, which is equivalent to taking $\mathrm{Gal}(F(\boldsymbol{\mu}_p)/F)$-invariant.

Definition 5.3. Define the semi-local Coleman map by setting

$$\mathrm{Col}_{\mathfrak{p}} := \bigoplus_{\mathfrak{q}|p} \mathrm{Col}_{p}^{\mathfrak{q}} : H^1(F_p, \mathbb{T}_{\mathrm{cyc}}) \longrightarrow R^{2g} .$$

For $1 \leq i \leq 2g$, we define

$$\mathrm{Col}_{\mathfrak{p},i} : H^1(F_p, \mathbb{T}_{\mathrm{cyc}}) \longrightarrow R$$

to be the projection of $\mathrm{Col}_{\mathfrak{p}}$ to the i-th component of R^{2g}.

Definition 5.4. Let $\mathfrak{I}$ be the set of subsets of $\{1, \ldots, 2g\}$ that are of size g. For any $I \in \mathfrak{I}$ we set

$$\mathrm{Col}_{\mathfrak{p}}^{I} := \bigoplus_{i \in I} \mathrm{Col}_{\mathfrak{p},i} : H^1(F_p, \mathbb{T}_{\mathrm{cyc}}) \longrightarrow R^g .$$

When no confusion may arise, we shall suppress I and $\mathfrak{p}$ from the notation and simply write $\mathfrak{C}$ in place of $\mathrm{Col}_{\mathfrak{p}}^{I}$.

Remark 5.5. By [BL14, Corollary 2.22], the image of $\mathrm{Col}_{\mathfrak{p}}^{I}$ is pseudo-isomorphic to a free R-module of rank g. Furthermore, if we choose a strongly admissible basis (see [BL14, Definition 3.2 and Proposition 3.3], then $\mathrm{Col}_{\mathfrak{p}}^{I}$ is in fact pseudo-surjective.

Definition 5.6. Let Z be a free R-submodule of rank g contained in the target R^g of the Coleman map $\mathrm{Col}_{\mathfrak{p}}^{I}$ and that contains the image of $\mathrm{Col}_{\mathfrak{p}}^{I}$ with finite index. The proof of [BL14, Corollary 2.22] shows that such Z exists.

Let $J_{\mathfrak{q}}$ denote the set of embeddings $F \hookrightarrow F_{\mathfrak{q}}$ for each $\mathfrak{q}|p$. Fix a bijection

$$\mathfrak{b} : \{1, \cdots, 2g\} \longrightarrow \coprod_{\mathfrak{q}|p} J_{\mathfrak{q}} .$$

For $I \in \mathfrak{I}$, set $I_{\mathfrak{q}} := \mathfrak{b}(I) \cap J_{\mathfrak{q}}$.

Definition 5.7. For each $I \in \mathfrak{I}$, we define

$$\mathbf{V}_I^{\mathrm{cyc}} := \bigcap_{i\in I} \ker \mathrm{Col}_{\mathfrak{p},i} = \ker \mathfrak{C}.$$

The orthogonal complement of $\mathbf{V}_I^{\mathrm{cyc}}$ under the pairing

$$H^1(F_p, T) \times H^1(F_p^{\mathrm{cyc}}, A^\vee[\mathfrak{p}^\infty]) \to \mathbb{Q}_p/\mathbb{Z}_p$$

is denoted by $\mathbf{V}_I^{\mathrm{cyc},\perp}$.

This allows us to define the I-Selmer group

$$\mathrm{Sel}_{\mathfrak{p}}^I(A^\vee/F^{\mathrm{cyc}}) := \ker\left(\mathrm{Sel}_{\mathfrak{p}}(A^\vee/F^{\mathrm{cyc}}) \longrightarrow \frac{H^1(F_p^{\mathrm{cyc}}, A^\vee[\mathfrak{p}^\infty])}{\mathbf{V}_I^{\mathrm{cyc},\perp}}\right). \tag{5.1}$$

As explained in [BL14, Appendix D], when A is an elliptic curve over $\mathbb{Q}$ with good supersingular reduction at p, on choosing an appropriate basis for the Dieudonné module, these Selmer groups coincide with Kobayashi's $\pm$-Selmer groups defined in [Kob03].

Remark 5.8. Note that our assumption (H.K.) ensures that A is supersingular at all primes above p by [Sug12]. In fact, we could define I-Selmer groups in the following way if A is ordinary at some primes above p. Let S_s be the set of primes above p where A has supersingular reduction and write $g_s = \sum_{q\in S_s}[F_{\mathfrak{q}} : \mathbb{Q}_p]$, which is even by Lemma 5.1. For each $\mathfrak{q} \in S_s$, we have the Coleman map

$$\mathrm{Col}_{\mathfrak{p}}^s : \oplus_{\mathfrak{q}\in S_s} H^1(F_{\mathfrak{q}}, \mathbb{T}_{\mathrm{cyc}}) \longrightarrow R^{g_s}$$

as above. For every subset I of $\{1, \ldots, g_s\}$ of size $g_s/2$, we can define V_I^{cyc} to be $\cap_{i\in I} \ker \mathrm{Col}_{\mathfrak{p},i}$ as before. Our Selmer group is then

$$\mathrm{Sel}_{\mathfrak{p}}^I(A^\vee/F^{\mathrm{cyc}}) := \ker\left(\mathrm{Sel}_{\mathfrak{p}}(A^\vee/F^{\mathrm{cyc}}) \longrightarrow \frac{\oplus_{\mathfrak{q}\in S_s} H^1(F_{\mathfrak{q}}^{\mathrm{cyc}}, A^\vee[\mathfrak{p}^\infty])}{\mathbf{V}_I^{\mathrm{cyc},\perp}}\right).$$

Lemma 5.9. *The R-module $\mathbf{V}_I^{\mathrm{cyc}}$ is free of rank g.*

Proof. Consider the following diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{V}_I^{\mathrm{cyc}} & \longrightarrow & H^1(F_p, \mathbb{T}_{\mathrm{cyc}}) & \xrightarrow{\mathfrak{C}} & Z \\
 & & \downarrow \mathrm{pr} & & \downarrow \mathrm{pr} & & \downarrow \mathrm{pr} \\
0 & \longrightarrow & \ker(\mathfrak{C}_F) & \longrightarrow & H^1(F_p, T) & \xrightarrow{\mathfrak{C}_F} & \overline{Z}
\end{array}$$

Here $\overline{Z} = Z \otimes \Lambda/\mathcal{A}_{\mathrm{cyc}}$ where $\mathcal{A}_{\mathrm{cyc}}$ is the augmentation ideal $\ker(R \to \mathfrak{O})$ and pr are the natural projections. Finally, the map $\mathfrak{C}_F$ is the map $\mathfrak{C}$ modulo $\mathcal{A}_{\mathrm{cyc}}$. Note that the middle vertical map is surjective by Proposition 3.3(i) and the last map is surjective by definition.

It follows from Remark 5.5 that the cokernel of $\mathfrak{C}_F$ is finite. Since $H^1(F_p, T)$ is free of rank $2g$, it follows that $\ker(\mathfrak{C}_F)$ is a free $\mathfrak{O}$-module of rank g. By Nakayama's lemma, this shows that the R-module $\mathbf{V}_I^{\mathrm{cyc}}$ may be generated by g elements. Using the fact that the R-module $\mathbf{V}_I^{\mathrm{cyc}}$ is torsion-free (being a submodule of the free R-module

$H^1(F_p, \mathbb{T}_{\mathrm{cyc}})$) one may easily prove that these generators cannot satisfy a non-trivial R-linear relation. $\square$

Choose a free Λ-module $\widetilde{Z}$ of rank g and an inclusion $\iota_Z : \widetilde{Z} \hookrightarrow \Lambda^g$ so that the diagram

$$\begin{array}{ccc} \widetilde{Z} & \xhookrightarrow{\iota_Z} & \Lambda^g \\ \otimes_\Lambda R \downarrow & & \downarrow \otimes_\Lambda R \\ Z & \xrightarrow[\subseteq]{} & R^g \end{array}$$

commutes. (This is possible by Nakayama's lemma.)

Proposition 5.10. *There is a (non-canonical) map* $\widetilde{\mathfrak{C}} : H^1(F_p, \mathbb{T}) \longrightarrow \widetilde{Z}$ *so that the following diagram commutes:*

$$\begin{array}{ccc} H^1(F_p, \mathbb{T}) & \xrightarrow{\widetilde{\mathfrak{C}}} & \widetilde{Z} \\ \downarrow & & \downarrow \\ H^1(F_p, \mathbb{T}_{\mathrm{cyc}}) & \xrightarrow{\mathfrak{C}} & Z \end{array}$$

where the (surjective) vertical arrows are induced from $\Gamma \twoheadrightarrow \Gamma_{\mathrm{cyc}}$.

We shall denote the composite map $H^1(F_p, \mathbb{T}) \xrightarrow{\widetilde{\mathfrak{C}}} \widetilde{Z} \xhookrightarrow{\iota_Z} \Lambda^g$ also by $\widetilde{\mathfrak{C}}$.

Proof. This follows from using the following observation iteratively: Let R be any local ring and let $I \subset R$ be any principal ideal. Let Y be a free R-module of finite rank. Then for any positive integer t, the kernel of the composition of the maps

$$\mathrm{Hom}_R\left(Y, R^t\right) \longrightarrow \mathrm{Hom}_R\left(Y, (R/I)^t\right) \xrightarrow{\sim} \mathrm{Hom}_{R/I}\left(Y/I, (R/I)^t\right)$$

is $I \cdot \mathrm{Hom}_R\left(Y, R^t\right)$. $\square$

Definition 5.11. Set $\mathbf{V}_I := \ker\left(\widetilde{\mathfrak{C}}\right)$. Recall the Dirichlet character $\rho = \omega_\psi$ and let $\mathcal{H}_I^1 \subset H^1(F_p, \mathbb{T}_\rho)$ be the Λ-submodule corresponding to $\mathbf{V}_I$ under the twisting isomorphism (3.2). For any finite extension $F \subset K \subset F_\infty$, let $\mathcal{H}_{I,K}^1$ denote the image of $\mathcal{H}_I^1$ under the (surjective) projection

$$H^1(F_p, \mathbb{T}_\rho) \longrightarrow H^1(K_p, T_\rho)\,.$$

Lemma 5.12. *The* Λ*-module* $\mathcal{H}_I^1$ *is free of rank* g.

Proof. Note that it suffices to verify the similar statement for $\mathbf{V}_I$. Proof of that, however, is identical to the proof of Lemma 5.9 and follows by considering the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{V}_I & \longrightarrow & H^1(F_p, \mathbb{T}) & \xrightarrow{\widetilde{\mathfrak{C}}} & \widetilde{Z} \\ & & \downarrow \mathrm{pr} & & \downarrow \mathrm{pr} & & \downarrow \mathrm{pr} \\ 0 & \longrightarrow & \mathbf{V}_I^{\mathrm{cyc}} & \longrightarrow & H^1(F_p, \mathbb{T}_{\mathrm{cyc}}) & \xrightarrow{\mathfrak{C}} & Z \end{array}$$

where pr now stands for reduction modulo the ideal $\ker(\Lambda \to R)$. $\square$

Definition 5.13. Let $\widetilde{\mathfrak{C}}^\rho$ denote the compositum of the maps

$$\widetilde{\mathfrak{C}}^\rho : H^1(F_p, \mathbb{T}_\rho) \xrightarrow{\text{tw}} H^1(F_p, \mathbb{T}) \xrightarrow{\widetilde{\mathfrak{C}}} \Lambda^g.$$

Remark 5.14. Note that we have $T \cong T_{\mathfrak{p}}(A)$ and $T^* \cong A^\vee[\varpi^\infty]$. Fix a prime v of F above p. Since we assumed that A is supersingular at v, it follows that the reduced variety has no p-torsion over any finite field of characteristic p. Thus the p-torsion subgroup of $A(F_v)$ should come from the formal group $\hat{A}$ of A. The formal group $\hat{A}$ is defined over an unramified extension F_v and thence the extension $F_v(\hat{A}[p])/F_v$ is totally ramified. By local class field theory, this shows that the hypothesis (1.2) holds true for $\chi = \rho$.

The hypothesis (1.3) holds true for ρ as the p-adic Hecke character ψ is surjective by our choice of the prime p, $f(\mathfrak{p}/p)$ is even by Lemma 5.1 and hence the order of $\rho = \omega_\psi$ is at least $p^2 - 1$.

6. Modified Selmer structures

In this section we introduce a variety of Selmer structures we shall analyze in what follows so as to prove the main conjectures for F_∞/F and the cyclotomic main conjectures for a CM abelian variety at a supersingular prime. We shall be working with two Galois representations (which will be ultimately related): First in Section 6.1 with the Galois representation $T_\chi \otimes \Lambda$ and later in Section 6.2 with $\mathbb{T}_\rho$ and $\mathbb{T}_{\text{cyc}}$. Those Selmer structures introduced in Section 6.2 will be used to obtain upper bounds for our Selmer groups, whereas those introduced in Section 6.1 will be used in order to prove the CM main conjecture and to prove (using the rigidity of Λ-adic Kolyvagin systems) that the upper bounds obtained for CM abelian varieties are indeed sharp in favourable situations.

6.1. Modified Selmer structures for Dirichlet characters.

In this section we shall be working with the Galois representation T_χ. The hypotheses (1.1), (1.2) and (1.3) on χ are in effect. Recall the collection $\mathfrak{E}$ of extensions of F, introduced in Section 1.2.

Definition 6.1. Let R be any ring and M be any R-module. For any submodule $N \subset M$, the *R-saturation of N in M* is the submodule $N^{\text{sat}} = \phi^{-1}\phi(N) \subset M$, where $\phi : M \to M \otimes \text{Frac}(R)$ is the natural map and $\text{Frac}(R)$ is the total ring of fractions of R.

Lemma 6.2. *The $\mathfrak{O}$-module $\mathcal{O}_L^{\times,\chi}$ is free of rank g.*

Proof. This follows from [NSW08, §8.6.12], along with our assumption that χ is different from the Teichmüller character ω. □

Definition 6.3.

(i) Let $\mathcal{B}_F := \text{loc}_p(\mathcal{O}_L^{\times,\chi})^{\text{sat}}$ be the $\mathfrak{O}$-saturation of $\text{loc}_p(\mathcal{O}_L^{\times,\chi})$ in $H^1(F_p, T_\chi)$. Note that the $\mathfrak{O}$-module $\mathcal{B}_F$ is a direct summand of the free module $H^1(M_p, T_\chi)$. Let the rank of the $\mathfrak{O}$-module $\mathcal{B}_F$ be $g - \mathfrak{d}$ with $\mathfrak{d} \geq 0$. Observe that $\mathfrak{d} = 0$ if Leopoldt's conjecture holds true for L.

(ii) Let $\mathcal{A}_F$ be any free submodule of $H^1(F_p, T)$ which complements $\mathcal{B}_F$.

(iii) Let $\mathcal{A}_{\mathfrak{K}}$ (resp., $\mathcal{B}_{\mathfrak{K}}$) be a direct summand of $\varprojlim_{M\in\mathfrak{C}} H^1(M_p, T_\chi)$ which maps onto $\mathcal{A}_F$ (resp., $\mathcal{B}_F$) under the natural (surjective) corestriction map. Note that such a direct summand exists by Nakayama's Lemma. Note further that we have the direct sum decomposition $\varprojlim_{M\in\mathfrak{C}} H^1(M_p, T_\chi) = \mathcal{A}_{\mathfrak{K}} \oplus \mathcal{B}_{\mathfrak{K}}$.

(iv) For $\mathcal{M} \in \mathfrak{C}$, let $\mathcal{A}_{\mathcal{M}} \subset H^1(\mathcal{M}_p, T_\chi)$ be the image of $\mathcal{A}_{\mathfrak{K}}$ under the natural projection and define similarly $\mathcal{B}_{\mathcal{M}}$.

(v) When $\mathcal{M} = F_{\text{cyc}}$, we write $\mathcal{A}_{\text{cyc}}$ in place of $\mathcal{A}_{F_{\text{cyc}}}$. We likewise define $\mathcal{B}_{\text{cyc}}$. When $\mathcal{M} = F_\infty$, we write $\mathcal{A}_\infty$ in place of $\mathcal{A}_{F_\infty}$ and define similarly $\mathcal{B}_\infty$.

Remark 6.4. If Leopoldt's conjecture holds true for L, then $\mathcal{B}_F$ is the unique direct summand of $H^1(F_p, T_\chi)$ of rank g, containing $\text{loc}_p(\mathcal{O}_L^{\times,\chi})$.

Proposition 6.5. *The intersection of $\mathcal{A}_{\text{cyc}}$ and $\text{loc}_p(H^1_{\mathcal{F}_{\text{can}}}(F, T_\chi \otimes \Lambda_{\text{cyc}}))$ is trivial. Likewise, the intersection of $\mathcal{A}_\infty$ and $\text{loc}_p(H^1_{\mathcal{F}_{\text{can}}}(F, \mathbb{T}_\chi))$ is trivial as well.*

Proof. We shall prove this proposition by induction on the Krull dimension of the relevant Iwasawa algebra. In the base case (when the coefficient ring in question is $\mathfrak{O}$ and its Krull dimension is 1), the assertion simply follows from the choice of $\mathcal{A}_F$. In order not to complicate further our notation, below we indicate the induction step from base case to Λ_{cyc}. Consider the following commutative diagram:

$$\begin{array}{ccc} H^1_{\mathcal{F}_{\text{can}}}(F, T_\chi \otimes \Lambda_{\text{cyc}}) & \xrightarrow{\ell_p^{\text{cyc}}} & H^1(F_p, T_\chi \otimes \Lambda_{\text{cyc}})/\mathcal{A}_{\text{cyc}} \\ \downarrow & & \downarrow \\ H^1_{\mathcal{F}_{\text{can}}}(F, T_\chi) & \overset{\ell_p}{\hookrightarrow} & H^1(F_p, T_\chi)/\mathcal{A}_F \end{array}$$

Suppose for some $\mathbb{U} \in H^1_{\mathcal{F}_{\text{can}}}(F, T \otimes \Lambda_{\text{cyc}})$ we have $\ell_p^{\text{cyc}}(\mathbb{U}) = 0$ and let U denote its image under the left vertical map. The diagram above shows that $U = 0$, thence

$$\mathbb{U} \in \ker\left(H^1_{\mathcal{F}_{\text{can}}}(F, T \otimes \Lambda_{\text{cyc}}) \to H^1(F, T)\right) = (\gamma_{\text{cyc}} - 1) H^1_{\mathcal{F}_{\text{can}}}(F, T \otimes \Lambda_{\text{cyc}}),$$

where γ_{cyc} is any topological generator of Γ^{cyc}. Write $\mathbb{U} = (\gamma_{\text{cyc}} - 1)\mathbb{U}_0$. We have therefore $(\gamma_{\text{cyc}} - 1)\ell_p^{\text{cyc}}(\mathbb{U}_0) = 0$ by the choice of $\mathbb{U}$. As $H^1(F_p, T \otimes \Lambda_{\text{cyc}})/\mathcal{A}_{\text{cyc}}$ is Λ_{cyc}-torsion free, it follows that $\ell_p^{\text{cyc}}(\mathbb{U}_0) = 0$, and repeating the argument above we conclude that $\mathbb{U}_0 = (\gamma_{\text{cyc}} - 1)\mathbb{U}_1$ with $\mathbb{U}_1 \in H^1_{\mathcal{F}_{\text{can}}}(F, T \otimes \Lambda_{\text{cyc}})$. On running this procedure k times, we conclude that $\mathbb{U} \in (\gamma_{\text{cyc}} - 1)^k H^1_{\mathcal{F}_{\text{can}}}(F, T \otimes \Lambda_{\text{cyc}})$ for every k and hence $\mathbb{U} = 0$. We conclude that the map ℓ_p^{cyc} is injective, completing the verification of the inductive step (and proving the first assertion). $\square$

Definition 6.6.

(i) Let $\mathfrak{L}$ be any free, rank-one $\mathfrak{O}[[\mathfrak{G}(\mathfrak{K})]]$-direct summand of $\varprojlim_{M\in\mathfrak{C}} H^1(M_p, T_\chi)$ such that
  - $\mathfrak{L}$ is not contained in $\mathcal{A}_{\mathfrak{K}}$,
  - $\mathfrak{L} + \mathcal{A}_{\mathfrak{K}}$ is also direct summand of $\varprojlim_{M\in\mathfrak{C}} H^1(M_p, T_\chi)$.

(ii) For $\mathcal{M} \in \mathfrak{C}$, let $\mathcal{L}_M \subset H^1(M_p, T_\chi)$ be the image of $\mathfrak{L}$ under the natural projection $\varprojlim_N H^1(N_p, T_\chi) \twoheadrightarrow H^1(\mathcal{M}_p, T_\chi)$.

(iii) We write $\mathfrak{L}_{\text{cyc}}$ (resp., $\mathfrak{L}_\infty$) in place of $\mathfrak{L}_{F_{\text{cyc}}}$ (resp., $\mathfrak{L}_{F_\infty}$).

We refer the reader to [MR04, §2.1] for a general definition of a Selmer structure. We define here two of them that shall be useful for our purposes.

Definition 6.7.

- The $\mathfrak{L}$-*restricted Selmer structure* $\mathcal{F}_{\mathcal{L}}$ on $\mathbb{T}_\chi$ is given by the local conditions
  - $H^1_{\mathcal{F}_{\mathcal{L}}}(F_\mathfrak{q},\mathbb{T}_\chi)=H^1_{\mathcal{F}_{\mathrm{can}}}(F_\mathfrak{q},\mathbb{T}_\chi)$ for every prime $\mathfrak{q}\nmid p$, and
  - $H^1_{\mathcal{F}_{\mathcal{L}}}(F_p,\mathbb{T}_\chi)=\mathcal{A}_\infty\oplus\mathfrak{L}_\infty$.
- The *unit-transversal-Selmer structure* $\mathcal{F}_{\mathrm{tr}}$ is given by the local conditions
  - $H^1_{\mathcal{F}_{\mathrm{tr}}}(F_\mathfrak{q},\mathbb{T}_\chi)=H^1_{\mathcal{F}_{\mathrm{can}}}(F_\mathfrak{q},\mathbb{T}_\chi)$ for every prime $\mathfrak{q}\nmid p$, and
  - $H^1_{\mathcal{F}_{\mathrm{tr}}}(F_p,\mathbb{T}_\chi)=\mathcal{A}_\infty$.

Any of the Selmer structures above *propagates* (in the sense of [MR04, Example 2.1.7]) to give rise to Selmer structures on any subquotient of $\mathbb{T}_\chi$. In this article, we will be mostly interested in the quotients T_χ or $T_\chi\otimes\Lambda_{\mathrm{cyc}}$. The propagation of a Selmer structure $\mathcal{F}$ to subquotients will still be denoted by the same symbol $\mathcal{F}$. Given a Selmer structure $\mathcal{F}$ on $\mathbb{T}_\chi$, one may also define the *dual Selmer structure* $\mathcal{F}^*$ on $\mathbb{T}^*_\chi$ using local Tate duality, as in [MR04, Definition 2.3.1].

Recall the finite set Σ of primes of F which consists of all primes that ramify in L/F, all archimedean primes of F and all primes of F above p. Let F_Σ denote the maximal extension of F contained in $\bar{F}$ which is unramified outside Σ and let G_Σ denote the Galois group $\mathrm{Gal}(F_\Sigma/F)$.

Definition 6.8. For $\mathcal{F}=\mathcal{F}_{\mathrm{can}},\mathcal{F}_{\mathfrak{L}}$, or $\mathcal{F}_{\mathrm{tr}}$ as above, we define the $\mathcal{F}$-*Selmer group* on the subquotient X of $\mathbb{T}_\chi$ by setting

$$H^1_{\mathcal{F}}(F,X)=\ker\left(H^1(G_\Sigma,X)\longrightarrow\bigoplus_{\mathfrak{q}\in\Sigma}H^1(F_\mathfrak{q},X)/H^1_{\mathcal{F}}(F_\mathfrak{q},X)\right).$$

6.2. **Selmer structures for the Tate module of a CM abelian variety.** Let $I\in\mathfrak{I}$, $\mathbf{V}_I^{\mathrm{cyc}}\subset H^1(F_p,\mathbb{T}_{\mathrm{cyc}})$ and $\mathbf{V}_I\subset H^1(F_p,\mathbb{T})$ be the free rank-g submodule defined as in Section 5. Assume throughout that Leopoldt's conjecture holds for L.

Set $\mathcal{U}^\rho_\infty=\mathrm{loc}_p\left(H^1_{\mathcal{F}_{\mathrm{can}}}(F,\mathbb{T}_\rho)\right)$. This is the image under loc_p of the (ρ-part of the) inverse limit of global units along F_∞/F. The Λ-module $\mathcal{U}^\rho_\infty\subset H^1(F_p,\mathbb{T})$ is a torsion-free submodule and it may be generated by at most g elements by Nakayama's lemma.

Proposition 6.9. *There is an* $I\in\mathfrak{I}$ *such that* $\mathrm{tw}(\mathcal{U}^\rho_\infty)\cap\mathbf{V}_I=0$.

Proof. It follows from [BL14, Lemma 3.28] that we may choose an $I\in\mathfrak{I}$ so that

$$\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^{\mathrm{cyc}}\cap\mathbf{V}_I^{\mathrm{cyc}}=0\,,$$

where $\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^{\mathrm{cyc}}\subset H^1(F_p,\mathbb{T}_{\mathrm{cyc}})$ is the projection of $\mathrm{tw}\,(\mathcal{U}^\rho_\infty)$ under the map

$$H^1(F_p,\mathbb{T})\twoheadrightarrow H^1(F_p,\mathbb{T}_{\mathrm{cyc}}).$$

We contend to prove the proposition for this particular I. Let $F_*\supset F_{\mathrm{cyc}}$ be a $\mathbb{Z}_p$-power extension. Set $\Gamma_*=\mathrm{Gal}(F_*/F)$ and $\Lambda_*=\mathfrak{O}[[\Gamma_*]]$. We will write $\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^*\subset H^1(F_p,T\otimes\Lambda_*)$ (resp., $\mathbf{V}_I^*$) for the image of $\mathrm{tw}\,(\mathcal{U}^\rho_\infty)$ (resp., $\mathbf{V}_I$) under the projection $H^1(F_p,\mathbb{T})\twoheadrightarrow H^1(F_p,T\otimes\Lambda_*)$. We will prove by induction on the Krull dimension of Λ_* that $\mathbf{V}_I^*\cap\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^*=0$, similar to the proof of Proposition 6.5.

When $\Lambda_* = \Lambda_{\text{cyc}}$, the assertion follows from the choice of I. Suppose the conclusion holds true for F_*/F, and suppose $F_\dagger/F_*$ is a $\mathbb{Z}_p$-extension. Similar to above define $\Lambda_\dagger$ and tw $(\mathcal{U}_\infty^\rho)^\dagger \in H^1(F_p, T \otimes \Lambda_\dagger)$. Let $\gamma_\dagger$ be a topological generator of Gal$(F_\dagger/F_*)$. Consider the following commutative diagram (where the injectivity of loc$_*$ is the induction hypothesis):

$$\begin{array}{ccc} H^1_{\mathcal{F}_{\text{can}}}(F, T\otimes\Lambda_\dagger) & \xrightarrow{\text{loc}_\dagger} & H^1(F_p, T\otimes\Lambda_\dagger)/\mathbf{V}_I^\dagger \\ \downarrow & & \downarrow \\ H^1_{\mathcal{F}_{\text{can}}}(F, T\otimes\Lambda_*) & \xhookrightarrow[\text{loc}_*]{} & H^1(F_p, T\otimes\Lambda_*)/\mathbf{V}_I^* \end{array}$$

Suppose for some $\mathbb{U} \in H^1_{\mathcal{F}_{\text{can}}}(F, T \otimes \Lambda_\dagger)$ we have $\text{loc}_\dagger(\mathbb{U}) = 0$ and let $U = 0$ denote its image under the left vertical map. The diagram above shows that $U = 0$, thence

$$\mathbb{U} \in \ker\left(H^1_{\mathcal{F}_{\text{can}}}(F, T\otimes\Lambda_\dagger) \to H^1(F, T\otimes\Lambda_*)\right) = (\gamma_\dagger - 1)H^1_{\mathcal{F}_{\text{can}}}(F, T\otimes\Lambda_\dagger, .$$

Write $\mathbb{U} = (\gamma_\dagger - 1)\mathbb{U}_0$. We have therefore have $(\gamma_\dagger - 1)\text{loc}_\dagger(\mathbb{U}_0) = 0$ by the choice of $\mathbb{U}$. As $H^1(F_p, T\otimes\Lambda_\dagger)/\mathbf{V}_I^\dagger$ is $\Lambda_\dagger$-torsion free, it follows that $\text{loc}_\dagger(\mathbb{U}_0) = 0$, and repeating the argument above we conclude that $\mathbb{U}_0 = (\gamma_\dagger - 1)\mathbb{U}_1$ with $\mathbb{U}_1 \in H^1_{\mathcal{F}_{\text{can}}}(F, T\otimes\Lambda_\dagger)$. On running this procedure k times, we conclude that $\mathbb{U} \in (\gamma_\dagger - 1)^k H^1_{\mathcal{F}_{\text{can}}}(F, T\otimes\Lambda_\dagger)$ for every k and hence $\mathbb{U} = 0$. We conclude that the map loc$_\dagger$ in upper row is injective, completing the verification of the inductive step. □

Fix until the end $I \in \mathfrak{I}$ such that tw $(\mathcal{U}_\infty^\rho)^{\text{cyc}} \cap \mathbf{V}_I^{\text{cyc}} = 0$ and tw $(\mathcal{U}_\infty^\rho) \cap \mathbf{V}_I = 0$.

Corollary 6.10. *We define* $\text{loc}_p^{\text{col}}$ *as the compositum of the maps*

$$\text{loc}_p^{\text{col}} : H^1_{\mathcal{F}_{\text{can}}}(F, \mathbb{T}_\rho) \xrightarrow{\text{loc}_p} H^1(F_p, \mathbb{T}_\rho) \xrightarrow{\widetilde{\mathfrak{C}}^\rho} \Lambda^g .$$

Then $\text{loc}_p^{\text{col}}$ *is injective and the element* $\text{loc}_p^{\text{col},\otimes g}(\mathfrak{S}_\infty^\rho) \in \Lambda$ *is non-zero.*

By a slight abuse, let $\text{loc}_p^{\text{col}}$ *also denote the maps*

$$\text{loc}_p^{\text{col}} : H^1_{\mathcal{F}_{\text{can}}}(F, \mathbb{T}) \xrightarrow{\text{loc}_p} H^1(F_p, \mathbb{T}) \xrightarrow{\widetilde{\mathfrak{C}}} \Lambda^g ,$$

and

$$\text{loc}_p^{\text{col}} : H^1_{\mathcal{F}_{\text{can}}}(F, \mathbb{T}_{\text{cyc}}) \xrightarrow{\text{loc}_p} H^1(F_p, \mathbb{T}_{\text{cyc}}) \xrightarrow{\mathfrak{C}} \Lambda^g_{\text{cyc}} .$$

Then $\text{loc}_p^{\text{col},\otimes g}(\text{tw}(\mathfrak{S}_\infty^\rho)^{\text{cyc}}) \in \Lambda_{\text{cyc}}$ *(and therefore, also* $\text{loc}_p^{\text{col},\otimes g}(\text{tw}(\mathfrak{S}_\infty^\rho)) \in \Lambda$*) is non-zero.*

Definition 6.11. Let $\mathbb{L} \subset \widetilde{Z}$ be *any* free direct summand of rank one and set $\widetilde{\mathbb{L}} := \widetilde{Z}/\mathbb{L}$. Set

$$H^1_{\mathbb{L}}(F_p, \mathbb{T}) := \ker\left(H^1(F_p, \mathbb{T}) \xrightarrow{\widetilde{\mathfrak{C}}} \widetilde{\mathbb{L}}\right) .$$

Throughout this paper, we will make use of the following *Selmer structures* on the G_F-representation $\mathbb{T}$ or via propagation, on its various quotients (such as T and $\mathbb{T}_{\text{cyc}}$). The first of these Selmer structures will be auxiliary, whereas the second is direct generalization of Kobayashi's signed Selmer groups to our setting. We further note that via the twisting isomorphism (3.2), all these Selmer structures also define a Selmer structure on $\mathbb{T}_\rho$ and its various subquotients.

Definition 6.12.

- The $\mathbb{L}$-*restricted Selmer structure* is given by the local conditions
  - $H^1_{\mathcal{F}_{\mathbb{L}}}(F_\mathfrak{q},\mathbb{T}) = H^1_{\mathcal{F}_{\mathrm{can}}}(F_\mathfrak{q},\mathbb{T})$ for every prime $\mathfrak{q}\nmid p$, and
  - $H^1_{\mathcal{F}_{\mathbb{L}}}(F_p,\mathbb{T}) = H^1_{\mathbb{L}}(F_p,\mathbb{T})$.
- The *I-signed Selmer structure $\mathcal{F}_I$* is given by the local conditions
  - $H^1_{\mathcal{F}_I}(F_\mathfrak{q},\mathbb{T}) = H^1_{\mathcal{F}_{\mathrm{can}}}(F_\mathfrak{q},\mathbb{T})$ for every prime $\mathfrak{q}\nmid p$, and
  - $H^1_{\mathcal{F}_I}(F_p,\mathbb{T}) = \mathbf{V}_I$.

6.3. **Comparison of Selmer groups.** Note for residual representations we have $\overline{T}_\chi = \overline{\mathbb{T}}_\chi = \boldsymbol{\mu}_p\otimes\chi^{-1}$ and $\overline{T}=\overline{\mathbb{T}}=A[\mathfrak{p}]$ as G_F-representations. In particular, when χ is chosen to be ω_E^{-1}, it follows from the definition of T and the discussion in Remark 2.2 that both these residual representations we consider are isomorphic.

Let k denote the residue field of $\mathfrak{O}$.

Lemma 6.13. *Assume the truth of Leopoldt's conjecture for the number field L. We have*

$$\dim_k H^1_{\mathcal{F}_{\mathrm{tr}}}(F,\overline{T}_\chi) = \dim_k H^1_{\mathcal{F}^*_{\mathrm{tr}}}(F,\overline{T}^*_\chi)\ ,\ \dim_k H^1_{\mathcal{F}_I}(F,\overline{T}) = \dim_k H^1_{\mathcal{F}^*_I}(F,\overline{T}^*) \tag{6.1}$$

and

$$\dim_k H^1_{\mathcal{F}_{\mathfrak{L}}}(F,\overline{T}_\chi) = \dim_k H^1_{\mathcal{F}^*_{\mathfrak{L}}}(F,\overline{T}^*_\chi)+1, \dim_k H^1_{\mathcal{F}_{\mathbb{L}}}(F,\overline{T}) = \dim_k H^1_{\mathcal{F}^*_{\mathbb{L}}}(F,\overline{T}^*)+1 \tag{6.2}$$

Proof. As explained in Example 4.9 we have $H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi)\cong\mathcal{O}_{L_\chi}^{\times,\chi}$ and this module is free of rank g under the running assumptions. On the other hand, $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,T^*)\cong \mathrm{CL}(L)^\chi$ is finite and it follows from the discussion in Section 5.2 of [MR04] that

$$\begin{aligned}\dim_k H^1_{\mathcal{F}_{\mathrm{can}}}(F,\overline{T}_\chi) - \dim_k H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,\overline{T}^*_\chi) &= \mathrm{rank}_{\mathfrak{O}}\, H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\chi) - \mathrm{corank}_{\mathfrak{O}}\, H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,T^*_\chi)\\ &= g\end{aligned} \tag{6.3}$$

Observe that we have by the choices we have made that

$$\dim_k H^1_{\mathcal{F}_{\mathrm{can}}}(F_p,\overline{T}_\chi) - \dim_k H^1_{\mathcal{F}_{\mathrm{tr}}}(F_p,\overline{T}_\chi) = g\,,$$
$$\dim_k H^1_{\mathcal{F}_{\mathrm{can}}}(F_p,\overline{T}_\chi) - \dim_k H^1_{\mathcal{F}_{\mathfrak{L}}}(F_p,\overline{T}_\chi) = g-1\ .$$

Using [Wil95, Proposition 1.6] we conclude that

$$\begin{aligned}(\dim_k H^1_{\mathcal{F}_{\mathrm{can}}}(F,\overline{T}_\chi) - \dim_k H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,\overline{T}^*_\chi)) - (\dim_k H^1_{\mathcal{F}_{\mathrm{tr}}}(F,\overline{T}_\chi) - \dim_k H^1_{\mathcal{F}^*_{\mathrm{tr}}}(F,\overline{T}^*_\chi))\\ = \dim_k H^1_{\mathcal{F}_{\mathrm{can}}}(F_p,\overline{T}_\chi) - \dim_k H^1_{\mathcal{F}_{\mathrm{tr}}}(F_p,\overline{T}_\chi) = g\end{aligned}$$

and

$$\begin{aligned}(\dim_k H^1_{\mathcal{F}_{\mathrm{can}}}(F,\overline{T}_\chi) - \dim_k H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,\overline{T}^*_\chi)) - (\dim_k H^1_{\mathcal{F}_{\mathfrak{L}}}(F,\overline{T}_\chi) - \dim_k H^1_{\mathcal{F}^*_{\mathfrak{L}}}(F,\overline{T}^*_\chi))\\ = \dim_k H^1_{\mathcal{F}_{\mathrm{can}}}(F_p,\overline{T}_\chi) - \dim_k H^1_{\mathcal{F}_{\mathfrak{L}}}(F_p,\overline{T}_\chi) = g-1\,.\end{aligned}$$

The portion concerning the Selmer groups for $\overline{T}_\chi$ follows from (6.3). Making use of [MR04, Theorem 5.2.15] in place of (6.3), the assertions on the Selmer groups for $\overline{T}$ are deduced in an identical way. □

Remark 6.14. It follows from Lemma 6.13 and [MR04, Corollary 4.5.2] that the k-vector space of Kolyvagin systems $\mathbf{KS}(\mathcal{F}_{\mathfrak{L}},\overline{T}_\chi)$ (resp., $\mathbf{KS}(\mathcal{F}_{\mathbb{L}},\overline{T})$) have dimension one. On the other hand, it follows from the main theorem of Section B that these *residual* Kolyvagin systems deform to $\mathbb{T}_\chi$ and $\mathbb{T}$ and that the Λ-module $\overline{\mathbf{KS}}(\mathcal{F}_{\mathfrak{L}},\mathbb{T}_\chi)$ (resp.,

$\overline{\mathbf{KS}}(\mathcal{F}_{\mathbb{L}}, \mathbb{T})$) is free of rank one. Elements of these modules (namely, Kolyvagin systems) are used to bound the characteristic ideal of $H^1_{\mathcal{F}^*_{\mathfrak{L}}}(F, \mathbb{T}^*_\chi)^\vee$ (resp., $H^1_{\mathcal{F}^*_{\mathbb{L}}}(F, \mathbb{T}^*)^\vee$). The generators of the module of Kolyvagin systems are characterized by the property that the bounds they give on the characteristic ideal of $H^1_{\mathcal{F}^*_{\mathfrak{L}}}(F, \mathbb{T}^*_\chi)^\vee$ (resp., of $H^1_{\mathcal{F}^*_{\mathbb{L}}}(F, \mathbb{T}^*)^\vee$) are sharp.

We will later use the (conjectural) Rubin-Stark elements to construct these Kolyvagin systems and exploit facts recalled above in order to verify the sharpness of the bounds we shall obtain on the signed (cyclotomic) Selmer groups for a CM abelian variety. This is one of the novelties in this article: In [PR04], the corresponding statement (Kobayashi's conjecture) was deduced from a (2-variable) CM main conjecture by a descent argument. For this reason, Pollack and Rubin had to utilize the non-existence of pseudo-null submodules of various Iwasawa modules. The analogous statements are not available in our context and our methods here are designed exactly to by-pass this issue.

Lemma 6.15. *Let* $\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^{\mathrm{cyc}} \subset \mathrm{loc}_p\left(H^1(F, \mathbb{T}_{\mathrm{cyc}})\right)$ *be as in the proof of Proposition 6.9. If* $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, \mathbb{T}^*_{\mathrm{cyc}})^\vee$ *is* Λ_{cyc}*-torsion then so is the quotient* $\mathrm{loc}_p\left(H^1(F, \mathbb{T}_{\mathrm{cyc}})\right)/\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^{\mathrm{cyc}}$.

Proof. We will proceed by induction and our notation will be very similar to the proof of Proposition 6.9.

Let $F_* \supset F_{\mathrm{cyc}}$ be a $\mathbb{Z}_p$-power extension. Set $\Gamma_* = \mathrm{Gal}(F_*/F)$ and $\Lambda_* = \mathfrak{O}[[\Gamma_*]]$. We will write $\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^* \subset H^1(F_p, T \otimes \Lambda_*)$ (resp., $\mathbf{V}^*_I$) for the image of $\mathrm{tw}\,(\mathcal{U}^\rho_\infty)$ (resp., $\mathbf{V}_I$) under the projection $H^1(F_p, \mathbb{T}) \twoheadrightarrow H^1(F_p, T \otimes \Lambda_*)$. We will prove by an (descending) inductive argument on the Krull dimension of Λ_* that $\mathrm{loc}_p\left(H^1(F, T \otimes \Lambda_*)\right)/\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^*$ is Λ_*-torsion.

In the base case (i.e., when $\Lambda_* = \Lambda$), the assertion is evident (since the quotient in question is trivial). Suppose the conclusion holds true for the $\mathbb{Z}_p$-power extension $F_\dagger/F$ containing F_{cyc}, and suppose $F_*/F_\dagger$ is a $\mathbb{Z}_p$-extension. Similar to above define $\Lambda_\dagger$ and $\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^\dagger \in H^1(F_p, T \otimes \Lambda_\dagger)$. Let γ_* be a topological generator of $\mathrm{Gal}(F_*/F_\dagger)$. As in the discussion of Remark 4.11 (particularly, using Nekovář's control theorem as in (4.1)) we have

$$\mathrm{coker}\left(\widetilde{H}^1_{f,\mathrm{Iw}}(F_\Sigma/F_*, T) \longrightarrow \widetilde{H}^1_{f,\mathrm{Iw}}(F_\Sigma/F_\dagger, T)\right) \cong \widetilde{H}^2_{f,\mathrm{Iw}}(F_\Sigma/F_*, T)[\gamma_* - 1] \tag{6.4}$$

We claim that this module is $\Lambda_\dagger$-torsion. We note that

$$\widetilde{H}^2_{f,\mathrm{Iw}}(F_\Sigma/F_*, T) \cong H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T \otimes \Lambda_*)^*)^\vee$$

by [Nek06, 8.9.6.2]. Furthermore,

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T \otimes \Lambda_*)^*)^\vee/(\gamma_* - 1) \cong \left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T \otimes \Lambda_*)^*)[\gamma_* - 1]\right)^\vee \cong H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T \otimes \Lambda_\dagger)^*)^\vee$$

where the second isomorphism is by [MR04, Lemma 3.5.3]. Since we assumed that $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, \mathbb{T}^*_{\mathrm{cyc}})^\vee$ is torsion, we once again use [MR04, Lemma 3.5.3] to conclude that the $\Lambda_\dagger$-module $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T \otimes \Lambda_*)^*)^\vee/(\gamma_* - 1)$ is torsion. It follows from Lemma 4.10 that

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(F, (T \otimes \Lambda_*)^*)^\vee[\gamma_* - 1] \cong \widetilde{H}^2_{f,\mathrm{Iw}}(F_\Sigma/F_*, T)[\gamma_* - 1]$$

is $\Lambda_\dagger$-torsion as well, as we have claimed.

The quotient $\mathrm{loc}_p\left(H^1(F, T\otimes\Lambda_\dagger)\right)/\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^\dagger$ is a homomorphic image of the quotient

$$H^1(F, T\otimes\Lambda_\dagger)/\mathrm{im}\left(H^1(F,T\otimes\Lambda_*)\right)\cong \mathrm{coker}\left(\widetilde{H}^1_{f,\mathrm{Iw}}(F_\Sigma/F_*,T)\longrightarrow \widetilde{H}^1_{f,\mathrm{Iw}}(F_\Sigma/F_\dagger,T)\right).$$

This completes the verification of the induction step. □

Proposition 6.16. *Assume that Leopoldt's conjecture holds for L. Then,*

$$H^1_{\mathcal{F}_{\mathrm{tr}}}(F,T_\chi)=H^1_{\mathcal{F}_{\mathrm{tr}}}(F,T_\chi\otimes\Lambda_{\mathrm{cyc}})=H^1_{\mathcal{F}_{\mathrm{tr}}}(F,\mathbb{T}_\chi)=0$$

and if $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,\mathbb{T}_{\mathrm{cyc}}(E)^*)^\vee$ is Λ_{cyc}-torsion,

$$H^1_{\mathcal{F}_I}(F,\mathbb{T}_{\mathrm{cyc}})=H^1_{\mathcal{F}_I}(F,\mathbb{T})=0\,.$$

Proof. The first group of assertions follow from the definitions.

The quotient $\mathrm{loc}_p\left(H^1(F,\mathbb{T}_{\mathrm{cyc}})\right)/\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^{\mathrm{cyc}}$ is a torsion Λ_{cyc}-module by Lemma 6.15. Since $\mathrm{tw}\,(\mathcal{U}^\rho_\infty)^{\mathrm{cyc}}\cap\mathbf{V}_I^{\mathrm{cyc}}=0$ by our very choice of $I\in\mathfrak{I}$ and since $H^1(F_p,\mathbb{T}_{\mathrm{cyc}})/\mathbf{V}_I^{\mathrm{cyc}}$ is torsion free, it follows that $\mathrm{loc}_p\left(H^1(F,\mathbb{T}_{\mathrm{cyc}})\right)\cap\mathbf{V}_I^{\mathrm{cyc}}=0$. This means

$$H^1_{\mathcal{F}_I}(F,\mathbb{T}_{\mathrm{cyc}}):=\ker\left(H^1(F,\mathbb{T}_{\mathrm{cyc}})\stackrel{\mathrm{loc}_p}{\longrightarrow}\mathbf{V}_I^{\mathrm{cyc}}\right)=0\,.$$

The proof that $H^1_{\mathcal{F}_I}(F,\mathbb{T})=0$ follows by induction using Nakayama's Lemma at each step. □

Remark 6.17. The statement that $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,\mathbb{T}^*_{\mathrm{cyc}})^\vee$ is Λ_{cyc}-torsion is a form of the weak Leopoldt conjecture for T. See Theorem 7.4(iii) below where we verify this statement assuming the Explicit Reciprocity Conjecture 4.18 for the Perrin-Riou-Stark elements.

Proposition 6.18. *Assume that Leopoldt's conjecture holds for L and the weak Leopoldt conjecture for T.*

(i) *For $X=T_\chi, T_\chi\otimes\Lambda_{\mathrm{cyc}},\mathbb{T}_\chi$ (resp., $\mathfrak{L}_X=\mathfrak{L}_F,\mathfrak{L}_{\mathrm{cyc}},\mathfrak{L}_\infty$) the following sequence is exact:*

$$0\longrightarrow H^1_{\mathcal{F}_{\mathfrak{L}}}(F,X)\stackrel{\mathrm{loc}_p}{\longrightarrow}\mathfrak{L}_X\longrightarrow H^1_{\mathcal{F}^*_{\mathrm{tr}}}(F,X^*)^\vee\longrightarrow H^1_{\mathcal{F}^*_{\mathfrak{L}}}(F,X^*)^\vee\longrightarrow 0.$$

(ii) *For $X=\mathbb{T}_{\mathrm{cyc}},\mathbb{T}$ and $\mathbf{V}_I^X=\mathbf{V}_I^{\mathrm{cyc}},\mathbf{V}_I$ the following sequence is exact:*

$$0\longrightarrow H^1_{\mathcal{F}_{\mathbb{L}}}(F,X)\stackrel{\mathrm{loc}_p}{\longrightarrow}H^1_{\mathcal{F}_{\mathbb{L}}}(F_p,X)/\mathbf{V}_I^X\longrightarrow H^1_{\mathcal{F}^*_I}(F,X^*)^\vee\longrightarrow H^1_{\mathcal{F}^*_{\mathbb{L}}}(F,X^*)^\vee\longrightarrow 0.$$

Proof. This follows from Poitou-Tate global duality, used along with Proposition 6.16. □

7. Perrin-Riou-Stark Kolyvagin systems and main conjectures

The goal in this section is to verify that the hypotheses of Appendix A hold true and apply the main results therein with the (conjectural) Rubin-Stark element Euler system of rank r in order to deduce the (g+1)-variable main conjecture in this setting. These results will be useful to us for deducing one of the main results of this article (Theorem 7.7), namely the *full* (i.e., not only a divisibility statement in the) signed main conjecture for CM abelian varieties.

Remark 7.1. The statements in this section are unfortunately conditional on the truth of Rubin-Stark conjectures. However, we prove in Appendix B that the Kolyvagin systems which these conjectural elements yield (following the recipe of Appendix A.2) do exist *unconditionally*. It is these Kolyvagin systems that we use to bound the relevant Selmer groups; however, in order to link our bounds with the L-values, the connection of these Kolyvagin systems with the Rubin-Stark elements is essential. It would be interesting to obtain unconditional consequences of this fact, such as when the CM field F is absolutely abelian.

7.1. **CM main conjectures in ($g+1$)-variables.** We assume the truth of the Perrin-Riou-Stark conjecture, the Leopoldt conjecture for L and the Explicit Reciprocity Conjecture 4.18. We also assume that the set S that appears in the definition of Rubin-Stark elements (see Section 4) contains no non-archimedean prime of F that splits in L/F.

Recall the Selmer structures on $\mathcal{F}_{\mathbb{L}}$ or $\mathcal{F}_I$ on $\mathbb{T}$ from Definition 6.12. By the twisting morphism (3.2) we have an induced Selmer structure on $\mathbb{T}_\rho$ (and on its various subquotients, such as T_ρ and $T_\rho \otimes \Lambda_{\rm cyc}$). We denote this Selmer structure also by $\mathcal{F}_{\mathbb{L}}$.

The first two assertions in Theorem 7.4 below is one of the main applications of the $\mathbb{L}$-restricted Perrin-Riou-Stark Kolyvagin systems we shall construct in Theorem A.11 below (which will be applied by setting $X = T_\rho$ and $\Psi = \widetilde{\mathfrak{C}}_{\mathfrak{p}}^\rho$ for (i) and $X = T$ and $\Psi = \widetilde{\mathfrak{C}}$ for (ii)). Recall the rank-g Euler system $\mathcal{C}^{(g)}_{\text{R-S}} := \{\varepsilon^\rho_{\mathcal{K}}\}_{\mathcal{K}\in\mathfrak{E}}$ of Rubin-Stark elements. On choosing $\Xi = \{\Xi_\eta\}_\eta$ as in §§A.1 and A.2 below we obtain an $\mathbb{L}$-restricted Kolyvagin system $\boldsymbol{\kappa}^\Xi \in \overline{\mathbf{KS}}(\mathbb{T}_\rho, \mathcal{F}_{\mathbb{L}}, \mathcal{P})$ whose initial term is given by an element denoted by

$$c^\Xi_{F_\infty} \in H^1(F, \mathbb{T}_\rho).$$

We remark that the element $c^\Xi_{F_\infty}$ corresponds to the element denoted by $c^\Xi_{\infty,1}$ in Section A.2 (with $X = T_\rho$ and $\Psi = \widetilde{\mathfrak{C}}_{\mathfrak{p}}^\rho$). Let $\mathrm{tw}(c^\Xi_{F_\infty})^{\rm cyc} \in H^1_{\mathcal{F}_{\mathbb{L}}}(F, \mathbb{T}_{\rm cyc})$ denote the image of $\mathrm{tw}(c^\Xi_{F_\infty}) \in H^1_{\mathcal{F}_{\mathbb{L}}}(F, \mathbb{T})$.

Definition 7.2. Let $\boldsymbol{\kappa}^{\rm PS} \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}})$ denote the Kolyvagin system obtained from the twisted Perrin-Riou-Stark elements via the descent procedure we develop in Section A.2 (applied with $X = T$). Let $\boldsymbol{\kappa}^{\rm PS,cyc} \in \overline{\mathbf{KS}}(\mathbb{T}_{\rm cyc}, \mathcal{F}_{\mathbb{L}})$ denote its image under the obvious map. Note that the initial term of $\boldsymbol{\kappa}^{\rm PS}$ is given by $\mathrm{tw}(c^\Xi_{F_\infty}) \in H^1(F, \mathbb{T})$ and the initial term of $\boldsymbol{\kappa}^{\rm PS,cyc}$ by $\mathrm{tw}(c^\Xi_{F_\infty})^{\rm cyc} \in H^1(F, \mathbb{T}_{\rm cyc})$, the projection of $\mathrm{tw}(c^\Xi_{F_\infty})$.

Definition 7.3. Let $\mathrm{Cl}(F_\infty)^\rho = \varprojlim_{M \subset LF_\infty} \mathrm{Cl}(M)^\rho$ and define $\mathrm{Cl}(F_{\rm cyc})^\rho$ similarly. We have the identifications (by class field theory) $\mathrm{Cl}(F_\infty)^\rho = H^1_{\mathcal{F}^*_{\rm can}}(F, \mathbb{T}^*_\rho)^\vee$ and $\mathrm{Cl}(F_{\rm cyc})^\rho = H^1_{\mathcal{F}^*_{\rm can}}(F, (T_\rho \otimes \Lambda_{\rm cyc})^*)^\vee$.

The final part of the following theorem is our first result towards the main conjecture for the maximal $\mathbb{Z}_p$-power extension F_∞ of the CM field F.

Theorem 7.4. *Let* $\mathcal{L} := H^1_{\mathcal{F}_{\mathbb{L}}}(F_p, \mathbb{T}_\rho)/\mathcal{H}^1_I$ *and* $\mathcal{V} := H^1(F_p, \mathbb{T}_\rho)/\mathcal{H}^1_I$.

(i) $\mathrm{tw}\left(c^\Xi_{F_\infty}\right)^{\rm cyc} \neq 0$, *the* $\Lambda_{\rm cyc}$*-module* $H^1_{\mathcal{F}^*_{\mathbb{L}}}(F, \mathbb{T}^*_{\rm cyc})^\vee$ *is torsion and*

$$\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(F, \mathbb{T}^*_{\rm cyc})^\vee\right) \mid \mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}}(F, \mathbb{T}_{\rm cyc})/\Lambda_{\rm cyc} \cdot \mathrm{tw}\left(c^\Xi_{F_\infty}\right)^{\rm cyc}\right).$$

(ii) $\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(F, \mathbb{T}^*_\rho)^\vee\right)$ *divides* $\mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}}(F, \mathbb{T}_\rho)/\Lambda \cdot c^\Xi_{F_\infty}\right)$.

(iii) *Weak Leopoldt conjecture for* T *holds true.*

(iv) $\mathrm{char}\left(\mathrm{Cl}(F_\infty)^\rho\right)$ *divides* $\mathrm{char}\left(\wedge^g H^1(F, \mathbb{T}_\rho)/\Lambda \cdot \mathfrak{S}^\rho_\infty\right)$.

Remark 7.5. We explain that all the characteristic ideals that appear in the assertion of Theorem 7.4 are non-zero. We shall prove (without relying on any of the conclusions that shall follow) that the element $\mathrm{tw}\left(c^{\Xi}_{F_\infty}\right)^{\mathrm{cyc}}$ is non-zero. On the other hand, $H^1_{\mathcal{F}_{\mathbb{L}}}(F,\mathbb{T}_{\mathrm{cyc}})$ injects into the torsion-free, rank-one Λ_{cyc}-module $H^1_{\mathcal{F}_{\mathbb{L}}}(F_p,\mathbb{T}_{\mathrm{cyc}})/\mathbf{V}_I^{\mathrm{cyc}}$ by Proposition 6.18(ii) and it therefore follows that $H^1_{\mathcal{F}_{\mathbb{L}}}(F,\mathbb{T}_{\mathrm{cyc}})/\Lambda_{\mathrm{cyc}}\cdot\mathrm{tw}\left(c^{\Xi}_{F_\infty}\right)^{\mathrm{cyc}}$ is torsion. Arguing similarly, we also check at once that $H^1_{\mathcal{F}_{\mathbb{L}}}(F,\mathbb{T}_\rho)/\Lambda\cdot c^{\Xi}_{F_\infty}$ is a torsion Λ-module. Furthermore, since we assumed Leopoldt's conjecture for L, it follows that the element $\mathfrak{S}^{\rho}_{\infty}$ is non-zero (c.f., the proof of [Rub96, Proposition 6.6(ii)]). We conclude using Lemma 4.11 that the Λ-module $\wedge^g H^1(F,\mathbb{T}_\rho)/\Lambda\cdot\mathfrak{S}^{\rho}_{\infty}$ is torsion.

Likewise, all the characteristic ideals which are present on the right-sides of the divisibilities (7.1) and (7.3)-(7.7) below are non-zero, since every single one of the corresponding module is either a quotient of a rank-r module by another rank-r submodule (where $r=1$ or g).

Proof of Theorem 7.4. The Explicit Reciprocity Conjecture 4.18 implies that $\mathrm{tw}\left(c^{\Xi}_{F_\infty}\right)^{\mathrm{cyc}}\neq 0$. The rest of (i) is deduced using the Kolyvagin system machinery applied for the Selmer structure $\mathcal{F}_{\mathbb{L}}$ on $\mathbb{T}_{\mathrm{cyc}}$ (see [MR04, Section 5.3]). Note that the hypotheses (H0) and (H1) of loc.cit. hold obviously true, whereas (H2) holds with $\tau=1$. The truth of hypothesis (H3) follows from Remark 5.14 and the truth of (H4) is also trivial (since $p>3$ from start). The second part of the theorem follows via an enhancement of the arguments of [MR04, Section 5.3] using the "dimension reduction trick" due to Ochiai, as given in [Och05, pp. 145]). The proof of (iii) is immediate from (i) since we have a tautological containment $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,\mathbb{T}^*_{\mathrm{cyc}})\subset H^1_{\mathcal{F}^*_{\mathbb{L}}}(F,\mathbb{T}^*_{\mathrm{cyc}})$.

We now prove (iv). On applying the twisting morphism tw on the exact sequence of Proposition 6.18(ii) (which applies thanks to (iii)) we have the following exact sequence:

$$0\longrightarrow H^1_{\mathcal{F}_{\mathbb{L}}}(F,\mathbb{T}_\rho)/\Lambda\cdot c^{\Xi}_{F_\infty}\stackrel{\mathrm{loc}_p}{\longrightarrow}\mathcal{L}/\Lambda\cdot\mathrm{loc}_p(c^{\Xi}_{F_\infty})\longrightarrow H^1_{\mathcal{F}^*_I}(F,\mathbb{T}^*_\rho)^\vee\longrightarrow H^1_{\mathcal{F}^*_{\mathbb{L}}}(F,\mathbb{T}^*_\rho)^\vee\longrightarrow 0.$$

It follows from (ii) that

$$\mathrm{char}\left(H^1_{\mathcal{F}^*_I}(F,\mathbb{T}^*_\rho)^\vee\right)\,\Big|\,\mathrm{char}\left(\mathcal{L}/\Lambda\cdot\mathrm{loc}_p\left(c^{\Xi}_{F_\infty}\right)\right)=\frac{\mathrm{char}\left(\mathbb{L}/\Lambda\cdot\mathrm{loc}_p^{\mathrm{col}}\left(c^{\Xi}_{F_\infty}\right)\right)}{\mathrm{char}\left(\mathbb{L}/\widetilde{\mathfrak{C}}^{\rho}(\mathcal{L})\right)} \tag{7.1}$$

The identification $H^1_{\mathcal{F}_{\mathrm{can}}}(F,\mathbb{T}^*_\rho)^\vee=\mathrm{Cl}(F_\infty)^\rho$ yields

$$0\longrightarrow H^1_{\mathcal{F}_{\mathrm{can}}}(F,\mathbb{T}_\rho)\stackrel{\mathrm{loc}_p}{\longrightarrow}\mathcal{V}\longrightarrow H^1_{\mathcal{F}^*_I}(F,\mathbb{T}^*_\rho)^\vee\longrightarrow\mathrm{Cl}(F_\infty)^\rho\longrightarrow 0\,. \tag{7.2}$$

Recall the module $\mathcal{U}^{\rho}_{\infty}$. By slight abuse, we denote its isomorphic image inside $\mathcal{V}$ also by $\mathcal{U}^{\rho}_{\infty}$. The displayed equation (7.2) together with (7.1) shows

$$\mathrm{char}\left(\mathrm{Cl}(F_\infty)^\rho\right) \,\Big|\, \frac{\mathrm{char}\left(\mathbb{L}/\Lambda\cdot \mathrm{loc}_p^{\mathrm{col}}\left(c^{\Xi}_{F_\infty}\right)\right)}{\mathrm{char}\left(\mathbb{L}/\mathfrak{C}^\rho(\mathcal{L})\right)\cdot \mathrm{char}\left(\mathcal{V}/\mathcal{U}^\rho_\infty\right)}$$

$$= \frac{\mathrm{char}\left(\wedge^g \widetilde{Z}\,/\,\Lambda\cdot \mathrm{loc}_p^{\mathrm{col},\otimes g}\left(\mathfrak{S}^\rho_\infty\right)\right)}{\mathrm{char}\left(\mathbb{L}/\mathfrak{C}^\rho(\mathcal{L})\right)\cdot \mathrm{char}\left(\mathcal{V}/\mathcal{U}^\rho_\infty\right)} \tag{7.3}$$

$$= \frac{\mathrm{char}\left(\wedge^g \widetilde{Z}\,/\,\Lambda\cdot \mathrm{loc}_p^{\mathrm{col},\otimes g}\left(\mathfrak{S}^\rho_\infty\right)\right)}{\mathrm{char}\left(\mathrm{coker}(\mathcal{L}\stackrel{\widetilde{\mathfrak{C}}^\rho}{\to}\mathbb{L})\right)\cdot \frac{\mathrm{char}\left(\widetilde{Z}/\widetilde{\mathfrak{C}}^\rho(\mathcal{U}^\rho_\infty)\right)}{\mathrm{char}(\mathrm{coker}(\mathcal{V}\stackrel{\widetilde{\mathfrak{C}}^\rho}{\to}\widetilde{Z}))}} \tag{7.4}$$

$$= \frac{\mathrm{char}\left(\wedge^g \widetilde{Z}\,/\,\Lambda\cdot \mathrm{loc}_p^{\mathrm{col},\otimes g}\left(\mathfrak{S}^\rho_\infty\right)\right)}{\mathrm{char}\left(\widetilde{Z}/\widetilde{\mathfrak{C}}^\rho(\mathcal{U}^\rho_\infty)\right)} \tag{7.5}$$

$$= \mathrm{char}\left(\wedge^g \widetilde{\mathfrak{C}}^\rho(\mathcal{U}^\rho_\infty)\,/\,\Lambda\cdot \mathrm{loc}_p^{\mathrm{col},\otimes g}\left(\mathfrak{S}^\rho_\infty\right)\right) \tag{7.6}$$

$$= \mathrm{char}\left(\wedge^g H^1(F,\mathbb{T}_\rho)\Big/\Lambda\cdot \mathfrak{S}^\rho_\infty\right). \tag{7.7}$$

We explain the non-obvious steps. (7.3) holds true thanks to the following diagram with commutative squares:

$$\begin{array}{ccccccccc}
\mathfrak{S}^\rho_\infty & \in & \wedge^g H^1(F,\mathbb{T}_\rho) & \xrightarrow{\mathrm{loc}_p^{\otimes g}} & \wedge^g H^1(F_p,\mathbb{T}_\rho) & \xrightarrow{\widetilde{\mathfrak{C}}^\rho} & \wedge^g \widetilde{Z} & \ni & \mathrm{loc}_p^{\mathrm{col},\otimes g}\left(\mathfrak{S}^\rho_\infty\right) \\
\big\downarrow & & \Xi_1\big\downarrow & & \Xi_1\big\downarrow & & \sim\big\downarrow\varphi_1 & & \big\downarrow \\
c^{\Xi}_{F_\infty} & \in & H^1_{\mathcal{F}_{\mathbb{L}}}(F,\mathbb{T}_\rho) & \xrightarrow{\mathrm{loc}_p} & H^1_{\mathcal{F}_{\mathbb{L}}}(F_p,\mathbb{T}_\rho) & \xrightarrow{\widetilde{\mathfrak{C}}^\rho} & \mathbb{L} & \ni & \mathrm{loc}_p^{\mathrm{col}}\left(c^{\Xi}_{F_\infty}\right)
\end{array}$$

Here φ_1 is the isomorphism of Proposition A.8(i) (with $\eta = 1$, $\widetilde{Z}$ chosen in place of Λ^g and $\mathbb{L} \subset \widetilde{Z}$ in place of $L \subset \Lambda^g$), the map Ξ_1 in the middle is the map of Proposition A.8(ii) and Ξ_1 on the left is induced by functoriality from the one in the middle. The steps (7.4) and (7.7) follow since $\widetilde{\mathfrak{C}}^\rho$ is injective on $\mathcal{V}$ and the steps (7.5) is because both modules $\mathrm{coker}(\mathcal{L} \stackrel{\widetilde{\mathfrak{C}}^\rho}{\to} \mathbb{L}) \hookrightarrow \mathrm{coker}(\mathcal{V} \stackrel{\widetilde{\mathfrak{C}}^\rho}{\to} \widetilde{Z})$ are pseudo-null by the construction of (normalized) Coleman maps. Finally, step (7.6) is because the Λ-module $\mathcal{U}^\rho_\infty$ (and therefore, also its isomorphic image $\widetilde{\mathfrak{C}}^\rho$) is free of rank g. The proof of the theorem is now complete. □

Remark 7.6. The proof above in fact may be modified slightly to prove that the following statement holds true for every intermediate $\mathbb{Z}_p$-power extension F_*/F that either contains F^{cyc} or else $F_* = F$. Let Λ_* be as in the proof of Proposition 6.9 (when $F_* = F$ we set $\Lambda_* = \mathfrak{O}$ and $\mathrm{char}_{\Lambda_*}(?) = \mathrm{Fitt}_{\mathfrak{O}}(?)$) and let $\mathfrak{S}^\rho_*$ denote the Perrin-Riou-Stark element for the tower F_*/F. We then prove that $\mathrm{char}_{\Lambda_*}\left(\mathrm{Cl}(F_*)^\rho\right)$ divides $\mathrm{char}_{\Lambda_*}\left(\wedge^g H^1(F,T_\rho\otimes\Lambda_*)/\Lambda\cdot\mathfrak{S}^\rho_*\right)$ (when $F_* = F$, write $H^1_{\mathcal{F}_{\mathrm{can}}}(F,T_\rho)$ in place of $H^1(F,T_\rho\otimes\Lambda_*)$).

Theorem 7.7. *We assume the truth of the Perrin-Riou-Stark Conjecture 4.14, the Leopoldt conjecture for L and the Explicit Reciprocity Conjecture 4.18. We also assume that the set S that appears in the definition of Rubin-Stark elements (see Section 4) contains no non-archimedean prime of F that splits in L/F.*

(i) $\#\mathrm{Cl}(L)^\rho = [\wedge^g \mathcal{O}_L^{\times,\rho} : \mathfrak{O}\cdot\varepsilon_F^\rho]$.
(ii) *The divisibility in the statement of Theorem 7.4(iv) may be upgraded to an equality and* $\mathrm{char}\,(\mathrm{Cl}(F_\infty)^\rho) = \mathrm{char}\left(\wedge^g H^1(F,\mathbb{T}_\rho)/\Lambda\cdot\mathfrak{S}_\infty^\rho\right)$.
(iii) $\mathrm{char}\left(H^1_{\mathcal{F}_I^*}(F,\mathbb{T}^*)^\vee\right)$ *is generated by* $\mathrm{loc}_p^{\mathrm{col},\otimes g}\,(\mathrm{tw}(\mathfrak{S}_\infty^\rho))$.
(iv) *The divisibility in the statement of Theorem 7.4(ii) may be upgraded to an equality and* $\mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}^*}(F,\mathbb{T}^*)^\vee\right) = \mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}}(F,\mathbb{T})/\Lambda\cdot\mathrm{tw}(c^{\Xi}_{F_\infty})\right)$.
(v) *The Perrin-Riou-Stark Kolyvagin system* $\boldsymbol{\kappa}^{\mathrm{PS}}$ *is primitive.*
(vi) *The cyclotomic Perrin-Riou-Stark Kolyvagin system* $\boldsymbol{\kappa}^{\mathrm{PS,cyc}}$ *is primitive.*
(vii) $\mathrm{char}\left(H^1_{\mathcal{F}_I^*}(F,\mathbb{T}^*_{\mathrm{cyc}})^\vee\right)$ *is generated by* $\mathrm{loc}_p^{\mathrm{col},\otimes g}\,(\mathrm{tw}(\mathfrak{S}_\infty^\rho)^{\mathrm{cyc}})$.

Remark 7.8. The statement of Theorem 7.7(i) is a form of Gras' conjecture for L. The Iwasawa module $H^1_{\mathcal{F}_I^*}(F,\mathbb{T}^*)^\vee$ which appears in (iii) should be compared to the module $\hat{X}$ of [Rub91, §11] and the assertion (iii) should be thought of as a generalization of Rubin's main conjecture [Rub91, Theorem 4.1(ii)]. We will soon convert the statement of (vii) to a signed main conjecture, comparing the characteristic ideal of the signed (cyclotomic) Selmer group to a signed p-adic L-function. We also note that the element $\mathrm{loc}_p^{\mathrm{col},\otimes g}\,(\mathrm{tw}(\mathfrak{S}_\infty^\rho)^{\mathrm{cyc}})$ (and therefore, also the element $\mathrm{loc}_p^{\mathrm{col},\otimes g}\,(\mathfrak{S}_\infty^\rho)$) is non-zero thanks to Corollary 6.10.

Proof of Theorem 7.7. The proof of (i) is identical to the proof of [Büy13c, Theorem 5.2(i)] (also, a minor alteration of the proof of Theorem 7.4(iv) combined with an analytic class number formula gives a proof of this assertion as well).

Let F_*/F be any $\mathbb{Z}_p$-power extension as in Remark 7.6, we also adapt the notation from that paragraph (and from the proof of Proposition 6.9). We shall prove by induction on the Krull dimension of Λ_* that

$$\mathrm{char}_{\Lambda_*}(\mathrm{Cl}(F_*)^\rho) = \mathrm{char}_{\Lambda_*}\left(\wedge^g H^1(F,T_\rho\otimes\Lambda_*)/\Lambda\cdot\mathfrak{S}_*^\rho\right).$$

Note that the base case (which is when $\Lambda_* = \mathfrak{O}$) is equivalent to the statement of (i). Let $F_\dagger \subset F_*$ be a $\mathbb{Z}_p$-power extension of F such that $\Gamma_*/\Gamma_\dagger \cong \mathbb{Z}_p$ and let $\gamma_* \in \Gamma_*/\Gamma_\dagger$ be any topological generator. To ease notation, we set $U_? := H^1(F,T_\rho\otimes\Lambda_?)$ for $? = *,\dagger$ and suppose that we have already proved

$$\mathrm{char}_{\Lambda_\dagger}(\mathrm{Cl}(F_\dagger)^\rho) = \mathrm{char}_{\Lambda_\dagger}\left(\wedge^g U_\dagger/\Lambda\cdot\mathfrak{S}_\dagger^\rho\right).$$

It follows from Lemma 4.10 (applied with $G=\Gamma_*, H=\Gamma_\dagger, \gamma_H=\gamma_*$ and $\pi_H=\pi_\dagger$) that

$$\begin{aligned}\pi_\dagger(\mathrm{char}_{\Lambda_*}(\mathrm{Cl}(F_*)^\rho))\cdot\mathrm{char}_{\Lambda_\dagger}(\mathrm{Cl}(F_*)^\rho[\gamma_*-1]) &= \mathrm{char}_{\Lambda_\dagger}(\mathrm{Cl}(F_*)^\rho/(\gamma_*-1)\mathrm{Cl}(F_*)^\rho)\\ &= \mathrm{char}_{\Lambda_\dagger}(\mathrm{Cl}(F_\dagger)^\rho) \quad (7.8)\\ &= \mathrm{char}_{\Lambda_\dagger}\left(\wedge^g U_\dagger/\Lambda\cdot\mathfrak{S}_\dagger^\rho\right), \quad (7.9)\end{aligned}$$

where (7.8) follows from [MR04, Lemma 3.5.3] used as in the proof of Lemma 6.15. Using Lemma 4.10 once again we have

$$\pi_\dagger\left(\mathrm{char}_{\Lambda_*}\left(\wedge^g U_*/\Lambda\cdot\mathfrak{S}_*^\rho\right)\right)=\pi_\dagger\left(\mathrm{char}_{\Lambda_\dagger}\left(\wedge^g U_*/\Lambda\cdot\mathfrak{S}_*^\rho\right)\right)\cdot\mathrm{char}_{\Lambda_\dagger}\left(\left(\wedge^g U_*/\Lambda\cdot\mathfrak{S}_*^\rho\right)[\gamma_*-1]\right)$$

$$=\frac{\mathrm{char}_{\Lambda_\dagger}\left(\wedge^g U_\dagger/\Lambda\cdot\mathfrak{S}_\dagger^\rho\right)}{\mathrm{char}_{\Lambda_\dagger}\left(\wedge^g U_\dagger/\wedge^g\ \mathrm{im}(U_*\to U_\dagger)\right)}$$

$$=\frac{\mathrm{char}_{\Lambda_\dagger}\left(\wedge^g U_\dagger/\Lambda\cdot\mathfrak{S}_\dagger^\rho\right)}{\mathrm{char}_{\Lambda_\dagger}\left(\mathrm{coker}(U_*\to U_\dagger)\right)} \tag{7.10}$$

$$=\frac{\mathrm{char}_{\Lambda_\dagger}\left(\wedge^g U_\dagger/\Lambda\cdot\mathfrak{S}_\dagger^\rho\right)}{\mathrm{char}_{\Lambda_\dagger}\left(\mathrm{Cl}(F_*)^\rho[\gamma_*-1]\right)}\,. \tag{7.11}$$

We explain the first equality. The Λ_*-module $\left(\wedge^g U_*/\Lambda\cdot\mathfrak{S}_*^\rho\right)[\gamma_*-1]$ is pseudo-null by Lemma 4.10(ii), but since the Λ_*-module $\wedge^g U_*$ has no non-zero pseudo-null submodules and $\Lambda\cdot\mathfrak{S}_*^\rho$ is free, the quotient $\wedge^g U_*/\Lambda\cdot\mathfrak{S}_*^\rho$ does not have any pseudo-null submodules. Thence $\left(\wedge^g U_*/\Lambda\cdot\mathfrak{S}_*^\rho\right)[\gamma_*-1]=0$ and $\mathrm{char}_{\Lambda^\dagger}\left(\left(\wedge^g U_*/\Lambda\cdot\mathfrak{S}_*^\rho\right)[\gamma_*-1]\right)=R$. The step (7.10) follows since both $\Lambda_\dagger$-modules $U_\dagger$ and $\mathrm{im}(U_*\to U_\dagger)$ are free; whereas (7.11) follows from (4.1). The induction step now follows combining (7.9) and (7.11) and this completes the proof of (ii).

Given (ii), one may trace back in the proof of Theorem 7.4 to see that the characteristic ideal of the Λ-module $\left(H^1_{\mathcal{F}_I^*}(F,\mathbb{T}_\rho^*)^\vee\right)$ is generated by $\mathrm{loc}_p^{\mathrm{col},\otimes g}\left(\mathfrak{S}_\infty^\rho\right)$. A formal twisting argument (c.f., [Rub00, Chapter 6]) proves (iii). One may proceed similarly to prove (iv) as well.

Now (v) follows from (iv) and (vi) from (v) (as a primitive Kolyvagin system $\boldsymbol{\kappa}$ is characterized by its property that its image $\overline{\boldsymbol{\kappa}}$ as a Kolyvagin system for the residual representation is non-zero). (viii) is then the restatement of Theorem A.14(iii). ☐

7.2. **The (analytic) signed p-adic L-function and the signed (cyclotomic) main conjecture.** Let $\boldsymbol{\xi}=\boldsymbol{\xi}_1\wedge\cdots\wedge\boldsymbol{\xi}_g\in\mathfrak{U}_\infty^\psi$ be a generator verifying Conjecture 4.18. In this section we borrow ideas from our companion article [BL14] in order to define signed p-adic L-function and study its various properties. For a completion F_λ of F at a prime above p, let

$$\mathcal{L}_T^{F_\lambda}:H^1_{\mathrm{Iw}}(F_\lambda,T)\longrightarrow\mathcal{H}_{\mathfrak{O}}\otimes_{\mathfrak{O}}\mathbb{D}_{F_\lambda}(T)$$

denote Perrin-Riou's regulator map from Appendix C. Set $\mathbb{D}_p(T):=\oplus_{\mathfrak{q}|p}\mathbb{D}_{F_\mathfrak{q}}(T)$ and

$$\mathcal{L}_T:=\oplus_{\mathfrak{q}|p}\,\mathcal{L}_T^{F_\mathfrak{q}}\ :\ H^1_{\mathrm{Iw}}(F_p,T)\longrightarrow\mathcal{H}_{\mathfrak{O}}\otimes_{\mathfrak{O}}\mathbb{D}_p(T)$$

denote the semi-local regulator map.

Definition 7.9. The (Perrin-Riou's) p-adic L-function $\mathcal{L}_\mathfrak{p}(A^\vee)$ is defined as the image of the regulator map on the twisted Rubin-Stark element $\boldsymbol{\xi}$:

$$\mathcal{L}_\mathfrak{p}(A^\vee):=\mathcal{L}_T^{\otimes g}\left(\mathrm{loc}_p(\boldsymbol{\xi})\right)\in\mathcal{H}_{\mathfrak{O}}\otimes_{\mathfrak{O}}\wedge^g\mathbb{D}_p(T)\,.$$

Remark 7.10. In this remark we explain why $\mathcal{L}_\mathfrak{p}(A^\vee)$ is deserved to be called a p-adic L-function. Indeed, it follows from [BL14, Proposition 3.11] that the element

$\mathcal{L}_{\mathfrak{p}}(A^\vee)$ may be characterized by the following interpolation property: For every even Dirichlet character θ of conductor p^n, we have

$$\theta\left(\mathcal{L}_{\mathfrak{p}}(A^\vee)\right) = \sum_{I\in\mathfrak{I}} \left(\frac{p^n}{\tau(\theta^{-1})}\right)^g L_{\{p\}}(\psi,\theta^{-1},1)\frac{\Omega^I_{\psi,p}}{\Omega^I_\psi}\cdot\varphi^n\left(\wedge_{i\in\underline{I}}\, v_i\right)$$

When θ is the trivial character,

$$\theta\left(\mathcal{L}_{\mathfrak{p}}(A^\vee)\right) = \sum_{I\in\mathfrak{I}}(1-\varphi)(1-p^{-1}\varphi^{-1})^{-1}L_{\{p\}}(\psi,1)\frac{\Omega^I_{\psi,p}}{\Omega^I_\psi}\cdot\left(\wedge_{i\in\underline{I}} v_i\right).$$

Here Ω^I_ψ (resp., $\Omega^I_{\psi,p}$) is a complex (resp., p-adic) period as in the statement of Conjecture 4.18.

Definition 7.11. For any $I\in\mathfrak{I}$ recall that we have the Coleman map $\mathfrak{C}$. We define the signed p-adic L-function $\mathcal{L}^I_{\mathfrak{p}}(A^\vee)$ by setting

$$\mathcal{L}^I_{\mathfrak{p}}(A^\vee) = \det\left(\mathfrak{C}^I_T(\boldsymbol{\xi}_i)\right).$$

Remark 7.12. As explained in [BL14, Lemma 3.16], there exists an integer $n(I)\geq 0$ such that $\det(\mathrm{Im}(\mathfrak{C}^I_T)) = (\gamma_{\mathrm{cyc}}-1)^{n(I)}\Lambda_{\mathrm{cyc}}$. In particular, $(\gamma_{\mathrm{cyc}}-1)^{n(I)}$ divides $\mathcal{L}^I_{\mathfrak{p}}(A^\vee)$. Note that $n(I)=0$ if the basis we choose to define the Coleman map is strongly divisible in the sense of [BL14, Definition 3.2].

Let $M_T\in\mathrm{GL}_{2g}\left(\mathcal{H}_{\mathfrak{O}}\right)$ denote the *logarithmic matrix* given as in [BL14, §2.3 and §3.1]. For every pair $I,J\in\mathfrak{I}$, let $M^{I,J}_T$ be the $g\times g$ the submatrix of M_T whose entries are the those indexed by the elements of $I\times J$. The following Theorem shows that that Perrin-Riou's p-adic L-function may be decomposed into an $\mathcal{H}_{\mathfrak{O}}$-linear combination of the (integral) signed p-adic L-functions $\mathcal{L}^I_{\mathfrak{p}}(A^\vee)$, justifying our choice of terminology.

Note that the p-adic L-functions defined this way are compatible with the definition of Pollack's $\pm$-p-adic L-functions defined in [Pol03]. See [BL14, Appendix D] for details.

Theorem 7.13. *There is a decomposition*

$$\mathcal{L}_{\mathfrak{p}}(A^\vee) = \sum_{I,J\in\mathfrak{I}}\det(M^{I,J}_T)\cdot\mathcal{L}^J_{\mathfrak{p}}(A^\vee)\cdot\wedge_{i\in I}v_i\,.$$

Proof. This is [BL14, Theorem 3.18]. □

Definition 7.14. Let $\mathcal{C}$ denote the matrix of $(1-\varphi)^{-1}(p\varphi-1)$ acting on $\mathbb{D}_p(T^\dagger)$ with respect to our fixed (dual) basis. Besides $I\in\mathfrak{I}$ we have fixed above, let $J\in\mathfrak{I}$ be another element and let $\mathcal{D}_{I,J}$ denote the (I,J) co-minor of $\mathcal{C}$.

The following interpolation formula for the signed p-adic L-function follows from [BL14, Proposition 3.18]:

Proposition 7.15. *For an even Dirichlet character θ modulo p we have*

$$\theta\left(\mathcal{L}^I_{\mathfrak{p}}(A^\vee)\right) = \begin{cases} L_{\{p\}}(\psi,1)\sum_{J\in\mathfrak{I}}\mathcal{D}_{I,J}\frac{\Omega^J_{\psi,p}}{\Omega^J_\psi} & \text{if } \theta \text{ is trivial,}\\ \frac{p^g}{\tau(\theta^{-1})^g}L_{\{p\}}(\psi,\theta^{-1},1)\frac{\Omega^I_{\psi,p}}{\Omega^I_\psi} & \text{otherwise.}\end{cases}$$

We are now ready to prove the *signed main conjecture* for CM abelian varieties at supersingular primes.

Theorem 7.16. *Suppose that the Perrin-Riou-Stark elements verify the Explicit Reciprocity Conjecture 4.18. Then*

$$\mathrm{char}\left(\mathrm{Sel}_{\mathfrak{p}}^{I}(A^\vee/F^{\mathrm{cyc}})^\vee\right) = \frac{\mathcal{L}_{\mathfrak{p}}^{I}(A^\vee)}{(\gamma_{\mathrm{cyc}}-1)^{n(I)}}\cdot\Lambda_{\mathrm{cyc}}\,.$$

Proof. It is easy to see (relying on a theorem of Lutz away from p) that

$$H^1_{\mathcal{F}_I^*}(F_p,\mathbb{T}^*_{\mathrm{cyc}}) = \mathrm{Sel}_{\mathfrak{p}}^{I}(A^\vee/F^{\mathrm{cyc}})\,.$$

The proof follows by Theorem 7.7(vii) and the definition of the signed p-adic L-function $\mathcal{L}_{\mathfrak{p}}^{I}(A^\vee)$. □

8. Descent and the Birch–Swinnerton-Dyer Conjecture

Let S' be the set of primes of F whose elements are the bad primes of T, the primes above p and the infinite primes. If $v\in S'$ with $v\nmid p$, we define

$$J_v^I(F_n)=\bigoplus_{w_n|v}H^1(F_{n,v},A^\vee[\mathfrak{p}^\infty])\quad\text{and}\quad J_v^I(F^{\mathrm{cyc}})=\varinjlim_n J_v(F_n),$$

where w_n runs through all the primes of F_n dividing v. We also define

$$J_p^I(F_n)=\frac{H^1(F_{n,p},A^\vee[\mathfrak{p}^\infty])}{\mathbf{V}_I^{n,\perp}}\quad\text{and}\quad J_p^I(F^{\mathrm{cyc}})=\frac{H^1(F_p^{\mathrm{cyc}},A^\vee[\mathfrak{p}^\infty])}{\mathbf{V}_I^{\mathrm{cyc},\perp}}.$$

Here, $\mathbf{V}_I^{n,\perp}$ is the orthogonal complement of the projection of $\mathbf{V}_I^{cyc}$ in $H^1(F_{n,p},T)$ under the pairing

$$H^1(F_{n,p},T)\times H^1(F_{n,p},A^\vee[\mathfrak{p}^\infty])\to\mathbb{Q}_p/\mathbb{Z}_p.$$

Lemma 8.1. *We have equality $V_I^{0,\perp}=H^1_f(F_p,A^\vee[\mathfrak{p}^\infty])$.*

Proof. This is equivalent to saying that the projection of $\mathbf{V}_I^{cyc}=\ker\mathfrak{C}$ in $H^1(F_p,T)$ is equal to $H^1_f(F_p,T)$.

By [BL14, Lemma 2.17], we have the inclusion

$$H^1_f(F_p,T)\subset(\ker\mathfrak{C})_{\Gamma^{\mathrm{cyc}}}.$$

But since $\mathfrak{C}$ is pseudo-surjective, $(\ker\mathfrak{C})_{\Gamma_{\mathrm{cyc}}}$ is a rank-g $\mathbb{Z}_p$-module. Subsequently, both $H^1_f(F_p,T)$ and $(\ker\mathfrak{C})_{\Gamma^{\mathrm{cyc}}}$ are saturated submodule of $H^1(F_p,T)$ of rank g, so they must coincide. □

We thus have a commutative diagram
(8.1)

$$\begin{array}{ccccccc}
0\to & \mathrm{Sel}_{\mathfrak{p}}^{I}(A^\vee/F^{\mathrm{cyc}})^{\Gamma_{\mathrm{cyc}}} & \to & H^1(G_{S'}(F^{\mathrm{cyc}}),A^\vee[\mathfrak{p}^\infty])^{\Gamma_{\mathrm{cyc}}} & \to & \bigoplus_{v\nmid p}J_v^I(F^{\mathrm{cyc}})^{\Gamma_{\mathrm{cyc}}}\oplus J_p^I(F^{\mathrm{cyc}})^{\Gamma_{\mathrm{cyc}}} \\
 & \uparrow\alpha & & \uparrow\beta & & \uparrow\gamma\oplus\gamma_p \\
0\to & \mathrm{Sel}_{\mathfrak{p}}(A^\vee/F) & \to & H^1(G_{S'}(F),A^\vee[\mathfrak{p}^\infty]) & \to & \bigoplus_{v\nmid p}J_v^I(F)\oplus J_p^I(F),
\end{array}$$

where the vertical maps are restriction maps.

Lemma 8.2. *The maps β, γ are isomorphisms and γ_p is a monomorphism.*

Proof. By Remark 5.14, $(A^\vee[\mathfrak{p}^\infty])^{\Gamma_{\rm cyc}} = 0$ and we conclude that the maps β and γ are isomorphisms by the inflation-restriction sequence.

As for the map γ_p, we consider the commutative diagram

$$\begin{array}{ccccc} H^1_f(F_p, A^\vee[\mathfrak{p}^\infty]) & \hookrightarrow & H^1(F_p, A^\vee[\mathfrak{p}^\infty]) & \twoheadrightarrow & H^1(F_p, A^\vee[\mathfrak{p}^\infty])/H^1_f(F_p, A^\vee[\mathfrak{p}^\infty]) \\ \downarrow & & \downarrow & & \downarrow \gamma_p \\ H^0\left(\Gamma_{\rm cyc}, \mathbf{V}_I^{\rm cyc,\perp}\right) & \hookrightarrow & H^0\left(\Gamma_{\rm cyc}, H^1(F_p^{\rm cyc}, A^\vee[\mathfrak{p}^\infty])\right) & \longrightarrow & H^0\left(\Gamma_{\rm cyc}, \frac{H^1(F_p^{\rm cyc}, A^\vee[\mathfrak{p}^\infty])}{\mathbf{V}_I^{\rm cyc,\perp}}\right). \end{array}$$

The left-most vertical map is an isomorphism by Lemma 8.1 and the vertical map in the middle is an isomorphism by the inflation-restriction sequence. It follows from Snake Lemma that the map γ_p is a monomorphism. □

Proposition 8.3. *The map*

$$\alpha:\ \mathrm{Sel}_{\mathfrak{p}}(A^\vee/F) \longrightarrow \mathrm{Sel}^I_{\mathfrak{p}}(A^\vee/F^{\rm cyc})^{\Gamma_{\rm cyc}}$$

is an isomorphism.

Proof. This follows chasing the diagram (8.1) with the aid of Lemma 8.2. □

Recall that we wrote $A \sim_p B$ for $A, B \in \overline{\mathbb{Q}}_p$ if $\mathrm{ord}_p(A/B) = 0$.

Theorem 8.4. *Assume that the hypotheses of Theorem 7.16 hold true and suppose that $I \in \mathfrak{I}$ is as in the proof of Proposition 6.9 with $n(I) = 0$. Then the following two assertions are equivalent:*

(i) $L_{\{p\}}(\psi, 1) \neq 0$ *and the p-adic period* $\sum_{J\in\mathfrak{I}} \mathcal{D}_{I,J} \frac{\Omega^J_{\psi,p}}{\Omega^J_\psi}$ *does not vanish.*

(ii) $\mathrm{Sel}_{\mathfrak{p}}(A^\vee/F)$ *is finite.*

In either case,

$$|\mathrm{Sel}_{\mathfrak{p}}(A^\vee/F)| \sim_p L_{\{p\}}(\psi, 1) \cdot \sum_{J\in\mathfrak{I}} \mathcal{D}_{I,J} \frac{\Omega^J_{\psi,p}}{\Omega^J_\psi} \ .$$

Proof. This follows easily from Theorem 7.16, Proposition 7.15 and Proposition 8.3. □

Acknowledgements. We would like to thank Henri Darmon, Eyal Goren, Frans Oort, Ken-ichi Sugiyama Chia-Fu Yu and Hui June Zhu for answering their questions during the preparation of this paper. We would also like to thank the anonymous referee for a number of invaluable suggestions and corrections on earlier versions of this paper.

APPENDIX A. COLEMAN-ADAPTED RANK-g EULER SYSTEM ARGUMENT

The goal of this appendix is to modify the rank-r Euler-Kolyvagin system machinery (developed in [Büy10]) so as to adapt them for our purposes in this article. The main difficulty is due to the fact that the Coleman maps do not in general have free images.

A.1. **Preliminaries.** Let Φ be a finite extension of $\mathbb{Q}_p$ and let $\mathfrak{O}$ be its ring of integers. Let k be either a totally real number field or a CM field, let k^{cyc} denote its cyclotomic $\mathbb{Z}_p$-extension and let k_∞ be its maximal $\mathbb{Z}_p$-power extension. Set $\Gamma = \mathrm{Gal}(k_\infty/k)$, $\Gamma_{\mathrm{cyc}} = \mathrm{Gal}(k^{\mathrm{cyc}}/k)$ and $\Lambda = \mathfrak{O}[[\Gamma]]$, $\Lambda_{\mathrm{cyc}} = \mathfrak{O}[[\Gamma^{\mathrm{cyc}}]]$. Set

$$\Gamma = \Gamma_{\mathrm{cyc}} \times \Gamma_1 \times \cdots \times \Gamma_r$$

and let $\gamma_?$ be a topological generator of $\Gamma_?$. (Note the slight difference here from the main text where Λ stood for $\mathbb{Z}_p[[\Gamma]]$ and $r = g + \mathfrak{d}$ in the main text.) As above for

$$\overline{m} = (m_{\mathrm{cyc}}, m_1, \cdots, m_r) \in (\mathbb{Z}_{\geq 0})^{r+1}$$

write

$$\Gamma_{\overline{m}} := \Gamma_{\mathrm{cyc}}/\Gamma_{\mathrm{cyc}}^{p^{m_{\mathrm{cyc}}}} \times \Gamma_1/\Gamma_1^{p^{m_1}} \times \cdots \times \Gamma_r/\Gamma_r^{p^{m_r}}$$

and let $k_{\overline{m}} \subset k_\infty$ the corresponding extension of k with Galois group $\Gamma_{\overline{m}}$.

Let X be a free $\mathfrak{O}$-module endowed with a continuous action of G_k. Set $X^* = \mathrm{Hom}(X, \boldsymbol{\mu}_{p^\infty})$ to denote the Cartier dual of X. Suppose that

(H.nA) For any place v of k above p that $H^0(k_v, X \otimes \Phi/\mathfrak{O}) = H^0(k_v, X^*) = 0$.

Let $r = \mathrm{rank}_{\mathfrak{O}} \mathrm{Ind}_{k/\mathbb{Q}} X)$ and $g := r - \mathrm{rank}_{\mathfrak{O}} H^0(\mathbb{R}, \mathrm{Ind}_{k/\mathbb{Q}} X)$.

Let $\mathcal{P}$ be the set of Kolyvagin primes for X and let $\mathcal{N} = \mathcal{N}(\mathcal{P})$ denote the square free products of primes chosen from $\mathcal{P}$. For $\mathfrak{q} \in \mathcal{P}$, let $k(\mathfrak{q})$ denote the maximal pro-p extension of k contained in the class field $k(\mathfrak{q})$ of conductor $\mathfrak{q}$. For $\eta = \mathfrak{q}_1 \cdots \mathfrak{q}_s$, let $k(\eta) := k(\mathfrak{q}_1) \cdots k(\mathfrak{q}_s)$. Set $\Delta_\eta = \mathrm{Gal}(k(\eta)/k)$ and $\Lambda(\eta) := \Lambda[\Delta_\eta] = \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta_\eta]$. Observe that

$$\Delta_\eta \cong \Delta_{\mathfrak{q}_1} \times \cdots \times \Delta_{\mathfrak{q}_s}.$$

Through this identification, we view Δ_μ (for $\mu \mid \eta$) as a subgroup of Δ_η. Note then that we may identify $\Delta_{\eta/\mu}$ with Δ_η/Δ_μ.

Lemma A.1. *The $\Lambda(\eta)$-module $H^1(k(\eta)_p, X \otimes \Lambda)$ (of the semi-local cohomology at p) is free of rank $2r$.*

Proof. This follows from Corollary 3.13 of [Büy10]. □

Lemma A.2. *There is a natural isomorphism*

$$\mathrm{Hom}_{\Lambda(\eta)}(H^1(k(\eta)_p, X \otimes \Lambda), \Lambda(\eta)) \xrightarrow{\sim} \mathrm{Hom}_{\Lambda}(H^1(k(\eta)_p, X \otimes \Lambda), \Lambda).$$

Proof. This is evident. The maps are given as follows:

$$\begin{array}{ccc} \mathrm{Hom}_{\Lambda(\eta)}(H^1(k(\eta)_p, X \otimes \Lambda), \Lambda(\eta)) & \longrightarrow & \mathrm{Hom}_{\Lambda}(H^1(k(\eta)_p, X \otimes \Lambda), \Lambda) \\ f = \left(x \mapsto f(x) = \sum_{\delta \in \Delta_\eta} f_\delta(x) \otimes \delta \in \Lambda \otimes \Delta_\eta\right) & \longmapsto & f_{\mathrm{id}} \\ \left(x \mapsto \sum_{\delta \in \Delta_\eta} f(x^\delta) \otimes \delta^{-1}\right) & \longleftarrow\!\shortmid & g \end{array}$$

□

Lemma A.3. *For $\mu \mid \eta$, the restriction map*

$$\mathrm{res}_{\eta/\mu} : H^1(k(\mu)_p, X \otimes \Lambda) \longrightarrow H^1(k(\eta)_p, X \otimes \Lambda)^{\Delta_{\eta/\mu}}$$

is an isomorphism.

Proof. Let v be any prime of $k(\eta)$ above p. Write $\mathcal{D}_v^\eta$ for the decomposition group of v inside Δ^η. We may identify $\mathcal{D}_v^\eta \subset \Delta^\eta$ with the local Galois group $\mathrm{Gal}(k(\eta)_v/k_\wp)$ where $\wp \subset k$ is the prime below v. If $\mathcal{D}_v^\eta$ is trivial, then $H^0(k(\eta)_v, X^*) = H^0(k_\wp, X^*) = 0$. If $\mathcal{D}_v^\eta$ is not trivial, then it is a non-trivial p-group, hence the order of $H^0\left(k(\eta)_v, X^*[p]\right)$ is congruent modulo p to the order of

$$H^0\left(k(\eta)_v, X^*[p]\right)^{\mathcal{D}_v^\eta} = H^0\left(k_\wp, X^*[p]\right) = 0,$$

thus $H^0\left(k(\eta)_v, X^*\right) = 0$ as well. The lemma now follows from the inflation -restriction sequence. □

Lemma A.4. *The quotient $\Lambda(\eta)/\Lambda(\eta)^{\Delta_{\eta/\mu}}$ is free as a $\Lambda(\mu)$-module.*

Proof. Note that the quotient $\mathbb{Z}_p[\Delta_{\eta/\mu}]/\mathbb{Z}_p[\Delta_{\eta/\mu}]^{\Delta_{\eta/\mu}}$ is a torsion-free, hence also a free $\mathbb{Z}_p$-module. As $\Lambda(\eta) = \Lambda(\mu) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta_{\eta/\mu}]$ and $\Lambda(\eta)^{\Delta_{\eta/\mu}} = \Lambda(\mu) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta_{\eta/\mu}]^{\Delta_{\eta/\mu}}$, the proof of the lemma follows. □

Lemma A.5. *The $\Lambda(\mu)$-module $H^1(k(\eta)_p, X \otimes \Lambda)/\mathrm{res}_{\eta/\mu}\left(H^1(k(\mu)_p, X \otimes \Lambda)\right)$ is free.*

Proof. This follows from Lemma A.1, A.3 and A.4. □

Lemma A.6. *For every $\mu \mid \eta$ and every $t \in \mathbb{Z}^+$, the map*

$$\mathrm{res}^*_{\eta/\mu} : \mathrm{Hom}(H^1(k(\eta)_p, X \otimes \Lambda), \Lambda(\eta)^t) \longrightarrow \mathrm{Hom}(H^1(k(\mu)_p, X \otimes \Lambda), \Lambda(\mu)^t)$$

is surjective.

Proof. Immediate from Lemma A.2 and A.5. □

Suppose now that we are given a Λ-module homomorphism

$$\Psi : \; H^1(k_p, X \otimes \Lambda) \longrightarrow \Lambda^g$$

with psuedo-null cokernel[5]. Using Lemma A.6 above, choose for each $\eta \in \mathcal{N}$ a homomorphism $\Psi^{(\eta)} \in \mathrm{Hom}(H^1(k(\eta)_p, X \otimes \Lambda), \Lambda(\eta)^g)$ which are compatible in the sense that the diagram

$$\begin{array}{ccc}
H^1(k(\eta)_p, X \otimes \Lambda) & \xrightarrow{\Psi^{(\eta)}} & \Lambda(\eta)^g \\
\uparrow & & \uparrow \\
H^1(k(\eta)_p, X \otimes \Lambda)^{\Delta_{\eta/\mu}} & & (\Lambda(\eta)^g)^{\Delta_{\eta/\mu}} \\
\uparrow \wr & & \uparrow \wr \\
H^1(k(\mu)_p, X \otimes \Lambda) & \xrightarrow{\Psi^{(\mu)}} & \Lambda(\mu)^g
\end{array}$$

commutes and such that $\Psi^{(1)} = \Psi$.

[5]In the main body of this article, Ψ will be an appropriate lift $\widetilde{\mathfrak{C}} : H^1(F_p, \mathbb{T}) \to \widetilde{Z}$ of a signed Coleman map, where we recall that $\widetilde{Z} \cong \Lambda^g$ contains the image of the signed Coleman map with pseudo-null cokernel. When we are dealing with the Galois representation $X = T_\rho$, we shall use the twisted Coleman map $\widetilde{\mathfrak{C}}^\rho$ in place of Ψ. See Section 5 for a detailed construction of these objects.

Fix a Λ-direct summand $L(1) = L$ of Λ^g of rank one and set $L(\eta) = L \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta_\eta]$. The $\Lambda(\eta)$-submodule $L(\eta) \subset \Lambda(\eta)^g$ is a direct summand of rank one and note for every $\mu \mid \eta$ that $L(\eta)^{\Delta_{\eta/\mu}} = L(\mu)$. We define

$$H^1_{\mathcal{F}_L}(k(\eta)_p, X \otimes \Lambda) := \ker \left(H^1(k(\eta)_p, X \otimes \Lambda) \xrightarrow{\Psi^{(\eta)}} \Lambda(\eta)^g / L(\eta) \right).$$

Definition A.7. For $\overline{m} = (m_{\rm cyc}, m_1, \cdots, m_r) \in (\mathbb{Z}_{\geq 0})^r$, let $H^1_{\mathcal{F}_L}(k_{\overline{m}}(\eta)_p, X)$ denote the image of $H^1_{\mathcal{F}_L}(k(\eta)_p, X \otimes \Lambda)$ inside of the free $\mathfrak{O}[\Delta_n \times \Gamma_m]$-module $H^1(k_{\overline{m}}(\eta)_p, X)$ of rank r.

Consider the maps

$$\text{(1.1)} \qquad \mathrm{Hom}_{\Lambda(\eta)}\left(\Lambda(\eta)^g / L(\eta), \Lambda(\eta)\right) \hookrightarrow \mathrm{Hom}_{\Lambda(\eta)}\left(\Lambda(\eta)^g, \Lambda(\eta)\right) \xrightarrow{\Psi^{(n)*}} \mathrm{Hom}_{\Lambda(\eta)}\left(H^1(k(\eta)_p, X \otimes \Lambda), \Lambda(\eta)\right)$$

Choose a basis $\{\psi_1^{(\eta)}, \cdots, \psi_{g-1}^{(\eta)}\}$ of the free $\Lambda(\eta)$-module $\mathrm{Hom}_{\Lambda(\eta)}\left(\Lambda(\eta)^g / L(\eta), \Lambda(\eta)\right)$, in a manner compatible with the variation in η. We then have an isomorphism

$$\bigoplus_{i=1}^{g-1} \psi_i^{(\eta)} : \Lambda(\eta)^g / L(\eta) \longrightarrow \Lambda(\eta)^{g-1}.$$

Let $\widetilde{\psi}_i^{(\eta)} \in \mathrm{Hom}_{\Lambda(\eta)}\left(\Lambda(\eta)^g, \Lambda(\eta)\right)$ under the first map in (1.1). Note that the map

$$\psi_\eta := \bigoplus_{i=1}^{g-1} \widetilde{\psi}_i^{(\eta)} : \Lambda(\eta)^g \longrightarrow \Lambda(\eta)^{g-1}$$

is surjective with kernel $L(\eta)$. Define

$$\varphi_\eta := \widetilde{\psi}_1^{(\eta)} \wedge \cdots \wedge \widetilde{\psi}_{g-1}^{(\eta)} \in \wedge^{g-1} \mathrm{Hom}_{\Lambda(\eta)}\left(\Lambda(\eta)^g, \Lambda(\eta)\right),$$

where the exterior product is taken in the category of $\Lambda(\eta)$-modules. Note that the collection $\{\varphi_\eta\}$ is compatible as η varies, by choice. Let

$$\Xi_\eta \in \wedge^{g-1} \mathrm{Hom}_{\Lambda(\eta)}\left(H^1(k(\eta)_p, X \otimes \Lambda), \Lambda(\eta)\right)$$

be the image of φ_η under the map induced from $\Psi^{(n)*}$.

Proposition A.8. (i) *For every $\eta \in \mathcal{N}$, φ_η maps $\wedge^g \Lambda(\eta)^g$ isomorphically onto $L(\eta)$.*
(ii) *For every $c \in \wedge^g H^1(k(\eta)_p, X \otimes \Lambda)$ we have $\Xi_\eta(c) \in H^1_{\mathcal{F}_L}(k(\eta)_p, X \otimes \Lambda)$.*

Proof. The proof of (i) is identical to that of [Büy10, Prop. 3.19]. For (ii), consider the following commutative diagram:

$$\begin{array}{ccccc} \wedge^g H^1(k(\eta)_p, X \otimes \Lambda) & \xrightarrow{\Xi_\eta} & H^1(k(\eta)_p, X \otimes \Lambda) & \supseteq & H^1_{\mathcal{F}_L}(k(\eta)_p, X \otimes \Lambda) \\ \downarrow{\scriptstyle \Psi^{(n)}} & & \downarrow{\scriptstyle \Psi^{(n)}} & & \downarrow{\scriptstyle \Psi^{(n)}} \\ \wedge^g \Lambda(\eta)^g & \xrightarrow[\varphi_\eta]{} & \Lambda(\eta)^g & \supseteq & L(\eta) \end{array}$$

where for the square in the right, recall that

$$H^1_{\mathcal{F}_L}(k(\eta)_p, X \otimes \Lambda) = (\Psi^\eta)^{-1}(L(\eta))$$

by definition. For $c \in \wedge^g H^1(k(\eta)_p, X \otimes \Lambda)$, we have $\varphi_\eta(\Psi^\eta(c)) \in L(\eta)$ by (i). By the commutativity of the diagram above this in turn means that $\Psi^\eta(\Xi_\eta(c)) \in L^{(\eta)}$, as desired. □

A.2. **Euler systems of rank r and Kolyvagin systems.** Suppose in this section that the hypotheses of Theorem 5.3.3 of [MR04] hold true. They hold in the setting of the main body of this article.

To ease notation, set

$$\Lambda_{\alpha,\overline{m}} := \Lambda/(\varpi^\alpha, \gamma_{\text{cyc}}^{p^{m_{\text{cyc}}}} - 1, \gamma_1^{p^{m_1}} - 1, \cdots, \gamma_r^{p^{m_r}} - 1)$$

for $\alpha \in \mathbb{Z}^+$ and $\overline{m} \in (\mathbb{Z}_{\geq 0})^{r+1}$.

Suppose we have an Euler system $\{c_{\overline{m},\eta}\}$ of rank r attached to the Galois representation X, in the sense of [Büy10, Definition 3.1] (so that $c_{\overline{m},\eta} \in \wedge^g H^1_{\mathcal{F}_\Lambda}(k_{\overline{m}}(\eta), X)$, where the exterior product is taken in the category of $\mathfrak{O}[\Gamma_{\overline{m}} \times \Delta_\eta]$-modules and $\mathcal{F}_\Lambda$ is the canonical Selmer structure defined as in [MR04, §5.3]). Following the recipe in [PR98, §1.2.3], one may than obtain an Euler system of rank 1 (or plainly, an Euler system in the sense of [Rub00]) $c^\Xi = \{c^\Xi_{\overline{m},\eta}\}$ attached to the collection $\Xi = \{\Xi_\eta\}$ above. Let

$$\boldsymbol{\kappa}^\Xi = \{\kappa_\eta^\Xi(\alpha, \overline{m}) \in H^1_{\mathcal{F}_\Lambda}(\eta)(k, X \otimes \Lambda_{\alpha,\overline{m}})\}_{\alpha,\overline{m},\eta} \in \overline{\mathbf{KS}}(X \otimes \Lambda, \mathcal{F}_\Lambda, \mathcal{P})$$

be the generalized Kolyvagin system attached to c^Ξ (for the Selmer structure $\mathcal{F}_\Lambda$ on $X \otimes \Lambda$, in the sense of [MR04, Definition 3.1.6]), which is the image of c_Ξ under the Euler systems to Kolyvagin systems map of [MR04, Theorem 3.2.4].

Remark A.9. The construction of the Euler systems to Kolyvagin systems map in [MR04] is given only when $\Lambda = \Lambda_{\text{cyc}}$. However their arguments extend in a trivial manner to the more general situation we consider here.

Definition A.10. We define the L-restricted Selmer structure $\mathcal{F}_L$ on $X \otimes \Lambda$ by setting

$$H^1_{\mathcal{F}_L}(k_\ell, X \otimes \Lambda) = H^1_{\mathcal{F}_\Lambda}(k_\ell, X \otimes \Lambda)$$

for $\ell \neq p$ and by letting

$$H^1_{\mathcal{F}_L}(k_p, X \otimes \Lambda) = \Psi^{-1}(L(1))$$

as above.

Theorem A.11. *The (generalized) Kolyvagin system $\boldsymbol{\kappa}^\Xi$ is in fact a (generalized) Kolyvagin system for the Selmer structure $\mathcal{F}_L$ on $X \otimes \Lambda$.*

Proof. We only need to verify that the classes $\kappa^\Xi_\eta(\alpha, \overline{m})$ verify the correct local condition at p, namely that

$$\text{loc}_p\left(\kappa^\Xi_\eta(\alpha, \overline{m})\right) \in H^1_{\mathcal{F}_L}(k_p, X \otimes \Lambda_{\alpha,\overline{m}})$$

for every $\alpha, \eta, \overline{m}$. Here $H^1_{\mathcal{F}_L}(k_p, X \otimes \Lambda_{\alpha,\overline{m}})$ stands for the image (under reduction mod $(\varpi^\alpha, \gamma_{\text{cyc}}^{p^{m_{\text{cyc}}}} - 1, \gamma_1^{p^{m_1}} - 1, \cdots, \gamma_r^{p^{m_r}} - 1)$) of $H^1_{\mathcal{F}_L}(k_p, X \otimes \Lambda)$.

Let $\widetilde{\kappa}_\eta(\alpha, \overline{m})^\Xi \in H^1_{\mathcal{F}^\eta_\Lambda}(k, X \otimes \Lambda_{\alpha,\overline{m}})$ be the class denoted by $\kappa_{[k_{\overline{m}},\eta,\alpha]}$ (the derivative class obtained from the Euler system c^Ξ) in [Rub00, Definition 4.4.10]. As in the proof of Theorems 3.25 and 3.29 of [Büy10], it suffices to verify that

$$\text{loc}_p\left(\widetilde{\kappa}_\eta(\alpha, \overline{m})^\Xi\right) \in H^1_{\mathcal{F}_L}(k_p, X \otimes \Lambda_{\alpha,\overline{m}})$$

for every $\eta \in \mathcal{N}$, every positive integer α and $\overline{m} \in (\mathbb{Z}_{\geq 0})^{r+1}$.

Let $L_{\overline{m}}(\eta) \subset \mathfrak{O}[\Gamma_{\overline{m}} \otimes \Delta_\eta]^g$ denote image of $L(\eta) \subset \Lambda(\eta)^g$. Note that $L_{\overline{m}}(\eta)$ is a free rank-1 direct summand of $\mathfrak{O}[\Gamma_{\overline{m}} \otimes \Delta_\eta]^g$.

Let $c^{\Xi}_{\infty,\eta} := \{c^{\Xi}_{\overline{m},\eta}\}_{\overline{m}} \in H^1(k(\eta), X \otimes \Lambda)$. By the defining property of Ξ we have $\Psi^\eta\left(\mathrm{loc}_p\left(c^{\Xi}_{\infty,\eta}\right)\right) \in L(\eta)$. Upon reduction modulo $(\varpi^\alpha, \gamma_{\mathrm{cyc}}^{p^{m_{\mathrm{cyc}}}} - 1, \gamma_1^{p^{m_1}} - 1, \cdots, \gamma_r^{p^{m_r}} - 1)$ we conclude that

$$\Psi^\eta\left(\mathrm{loc}_p\left(c^{\Xi}_{\overline{m},\eta}\right)\right) \in L_{\overline{m}}(\eta)$$

for every η and $\overline{m}$.

Let $D_\eta \in \mathfrak{O}[\Delta_\eta]$ be the derivative operator defined as in [Rub00, §4]. As the maps $\Phi^{(\eta)}$ and loc_p are Δ_η-equivariant, it follows that

$$\Psi^\eta\left(\mathrm{loc}_p\left(D_\eta c^{\Xi}_{\overline{m},\eta}\right)\right) \in L_{\overline{m}}(\eta)$$

as well. On reduction mod ϖ^α we see that

$$\Psi^\eta\left(\mathrm{loc}_p\left(D_\eta c^{\Xi}_{\overline{m},\eta} \bmod \varpi^\alpha\right)\right) \in L_{\overline{m}}(\eta)/\varpi^\alpha \cdot L_{\overline{m}}(\eta) \tag{1.2}$$

The fundamental property of the derivative operator D_η is that we have

$$D_\eta c^{\Xi}_{\overline{m},\eta} \bmod \varpi^\alpha \in H^1_{\mathcal{F}^\eta_\Lambda}(k(\eta), X \otimes \Lambda_{\alpha,\overline{m}})^{\Delta_\eta}.$$

Combining this fact with (1.2) we conclude that

$$\Psi^\eta\left(\mathrm{loc}_p\left(D_\eta c^{\Xi}_{\overline{m},\eta} \bmod \varpi^\alpha\right)\right) \in \left(L_{\overline{m}}(\eta)/\varpi^\alpha \cdot L_{\overline{m}}(\eta)\right)^{\Delta_\eta}. \tag{1.3}$$

By definition, the element $\widetilde{\kappa}_\eta(\alpha, \overline{m})^{\Xi}$ is a canonical inverse image of $D_\eta c^{\Xi}_{\overline{m},\eta} \bmod \varpi^\alpha$ under the restriction map

$$\mathrm{res} : H^1_{\mathcal{F}^\eta_\Lambda}(k, X \otimes \Lambda_{\alpha,\overline{m}}) \longrightarrow H^1_{\mathcal{F}^\eta_\Lambda}(k(\eta), X \otimes \Lambda_{\alpha,\overline{m}})^{\Delta_\eta}.$$

Thus,

$$\mathrm{res}\left(\mathrm{loc}_p\left(\widetilde{\kappa}_\eta(\alpha, \overline{m})^{\Xi}\right)\right) = \mathrm{loc}_p\left(D_\eta c^{\Xi}_{\overline{m},\eta} \bmod \varpi^\alpha\right). \tag{1.4}$$

Consider the following commutative diagram:

$$\begin{array}{ccccc}
H^1(k(\eta)_p, X \otimes \Lambda_{\alpha,\overline{m}})^{\Delta_\eta} & \xrightarrow{\Psi^{(\eta)}} & (\mathfrak{O}/\varpi^\alpha\mathfrak{O}\,[\Gamma_{\overline{m}} \otimes \Delta_\eta]^g)^{\Delta_\eta} & \supseteq & \left(L_{\overline{m}}(\eta)/\varpi^\alpha \cdot L_{\overline{m}}(\eta)\right)^{\Delta_\eta} \\
\uparrow{\scriptstyle \mathrm{res}} & & \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim} \\
H^1(k_p, X \otimes \Lambda_{\alpha,\overline{m}}) & \xrightarrow{\Psi} & \mathfrak{O}/\varpi^\alpha\mathfrak{O}\,[\Gamma_{\overline{m}}]^g & \supseteq & L_{\overline{m}}(1)/\varpi^\alpha \cdot L_{\overline{m}}(1)
\end{array}$$

where the isomorphism on the very right is because the $\mathfrak{O}/\varpi^\alpha\mathfrak{O}\,[\Gamma_{\overline{m}} \otimes \Delta_\eta]$-module $L_{\overline{m}}(\eta)/\varpi^\alpha \cdot L_{\overline{m}}(\eta)$ is free. By the commutativity of this diagram along with (1.2) and (1.4) it follows that

$$\Psi\left(\mathrm{loc}_p\left(\widetilde{\kappa}_\eta(\alpha, \overline{m})^{\Xi}\right)\right) \in L_{\overline{m}}(1)/\varpi^\alpha \cdot L_{\overline{m}}(1)$$

which is precisely to say that

$$\mathrm{loc}_p\left(\widetilde{\kappa}_\eta(\alpha, \overline{m})^{\Xi}\right) \in H^1_{\mathcal{F}_L}(k_p, X \otimes \Lambda_{\alpha,\overline{m}})$$

as desired. □

A.3. **Applications.** Suppose throughout this section that:

(H.dR) X is de Rham at every place v of k above p.
(H.wL) The cohomology class $c_{\infty,1}^{\Xi} \in H^1(k, X \otimes \Lambda)$ is non-trivial.

The second hypotheses implies by [Rub00, Theorem 2.3.2] that the weak Leopoldt conjecture holds true for X:

Theorem A.12. *Under the running hypotheses the Λ-module $H^1_{\mathcal{F}^*_\Lambda}(k, (X \otimes \Lambda)^*)^\vee$ is torsion.*

This assertion is equivalent by the Poitou-Tate global duality to the following:

Corollary A.13. *The Λ-module $H^1_{\mathcal{F}_\Lambda}(k, X \otimes \Lambda)$ has rank g.*

Consider the following hypothesis:

(H.V) The Ψ-*strict Selmer group*

$$H^1_{\mathcal{F}_\Psi}(k, X \otimes \Lambda) := \ker\left(H^1_{\mathcal{F}_L}(k, X \otimes \Lambda) \stackrel{\Psi \circ \mathrm{loc}_p}{\longrightarrow} \Lambda^g\right)$$

is trivial.

In specific applications this hypothesis will hold true for a certain Selmer group determined by a choice of a *signed* Coleman map in place of Ψ. Let $\mathcal{F}^*_\Psi$ denote the dual Selmer structure on $(X \otimes \Lambda)^*$ defined as in [MR04, Definition 1.3.1].

Write $c_{\overline{m},1} = c^{(1)}_{\overline{m}} \wedge \cdots \wedge c^{(g)}_{\overline{m}} \in \wedge^g H^1_{\mathcal{F}_\Lambda}(k_{\overline{m}}, X)$ and define

$$\begin{aligned}\mathfrak{c}_\infty := \mathfrak{c}_1 \wedge \cdots \wedge \mathfrak{c}_g := \{\mathrm{loc}_p\left(c^{(1)}_{\overline{m}}\right) \wedge \cdots \wedge \mathrm{loc}_p\left(c^{(1)}_{\overline{m}}\right)\}_{\overline{m}} &\in \varprojlim_{\overline{m}} \wedge^g H^1(k_{\overline{m},p}, X) \\ &= \wedge^g H^1(k_p, X \otimes \Lambda)\end{aligned}$$

where the last equality holds true thanks to our running hypothesis (H.nA) (from which follows that the Λ-module $H^1(k_p, X \otimes \Lambda)$ is free, the map $H^1(k_p, X \otimes \Lambda) \to H^1(k_{\overline{m},p}, X)$ is surjective and therefore the $\mathfrak{O}[\Gamma_{\overline{m}}]$-module $H^1(k_{\overline{m},p}, X)$ is free.)

Theorem A.14 (The Ψ-main conjecture)**.** *Under our running hypotheses,*

(i) *the Λ-module $H^1_{\mathcal{F}^*_\Psi}(k, (X \otimes \Lambda)^*)^\vee$ is torsion and*

$$\det\left([\Psi(\mathfrak{c}_i)]_{i=1}^g\right) \in \mathrm{char}\left(H^1_{\mathcal{F}^*_\Psi}(k, (X \otimes \Lambda)^*)^\vee\right),$$

(ii) *let $\mathfrak{c}_{\mathrm{cyc}} = \mathfrak{c}_1^{\mathrm{cyc}} \wedge \cdots \wedge \mathfrak{c}_g^{\mathrm{cyc}} \in \wedge^g H^1(k_p, X \otimes \Lambda_{\mathrm{cyc}})$ denote the image $\mathfrak{c}$. Then the characteristic ideal of the Λ_{cyc}-module $H^1_{\mathcal{F}^*_\Psi}(k, (X \otimes \Lambda_{\mathrm{cyc}})^*)^\vee$ contains* $\det\left([\Psi(\mathfrak{c}_i^{\mathrm{cyc}})]_{i=1}^g\right)$.

(iii) *If further the associated Kolyvagin system $\boldsymbol{\kappa}^\Xi$ is primitive, then the characteristic ideal of the Λ_{cyc}-module $H^1_{\mathcal{F}^*_\Psi}(k, (X \otimes \Lambda_{\mathrm{cyc}})^*)^\vee$ is generated by* $\det\left([\Psi(\mathfrak{c}_i^{\mathrm{cyc}})]_{i=1}^g\right)$.

Proof. By (H.V) it follows that the map

$$H^1_{\mathcal{F}_L}(k, X \otimes \Lambda) \stackrel{\Psi \circ \mathrm{loc}_p}{\longrightarrow} L(1)$$

is injective. We therefore conclude by our assumption (H.wL) that the quotient

$$H^1_{\mathcal{F}_L}(k, X \otimes \Lambda)/\Lambda \cdot c_{\infty,1}^{\Xi}$$

is Λ-torsion. As a consequence of Theorem A.11 (and applying the Kolyvagin system machinery (via an enhancement of [MR04, Theorem 5.3.10] using the "dimension reduction trick" due to Ochiai, as given in [Och05, pp. 145]) for the Kolyvagin system $\boldsymbol{\kappa}^{\Xi}$ for the Selmer structure $\mathcal{F}_L$, whose initial term is by definition $c^{\Xi}_{\infty,1}$) we see that

$$\text{(1.5)} \qquad \mathrm{char}\left(H^1_{\mathcal{F}_L^*}(k,(X\otimes\Lambda)^*)^\vee\right) \mid \mathrm{char}\left(H^1_{\mathcal{F}_L}(k,X\otimes\Lambda)/\Lambda\cdot c^{\Xi}_{\infty,1}\right)\neq 0\,.$$

Poitou-Tate global duality yields an exact sequence

$$0\to H^1_{\mathcal{F}_L}(k,X\otimes\Lambda)/\Lambda\cdot c^{\Xi}_{\infty,1}\longrightarrow L/\Lambda\cdot\Psi\circ\mathrm{loc}_p\left(c^{\Xi}_{\infty,1}\right)\longrightarrow H^1_{\mathcal{F}_\Psi^*}(k,(X\otimes\Lambda)^*)^\vee \longrightarrow H^1_{\mathcal{F}_L^*}(k,(X\otimes\Lambda)^*)^\vee\to 0$$

Upon taking characteristic polynomials and using (1.5) we conclude that

$$\text{(1.6)} \qquad \mathrm{char}\left(H^1_{\mathcal{F}_\Psi^*}(k,(X\otimes\Lambda)^*)^\vee\right) \mid \mathrm{char}\left(L/\Lambda\cdot\Psi\circ\mathrm{loc}_p\left(c^{\Xi}_{\infty,1}\right)\right)$$

By Proposition A.8 and the choice of Ξ, we have an isomorphism

$$\left(\wedge^g\Lambda^g\right)/\Lambda\cdot\left(\Psi(\mathfrak{c}_1)\wedge\cdots\wedge\Psi(\mathfrak{c}_g)\right)\stackrel{\varphi}{\longrightarrow} L/\Lambda\cdot\Psi\circ\mathrm{loc}_p\left(c^{\Xi}_{\infty,1}\right).$$

The proof of (i) follows. In fact the same proof (without appealing to the work of Ochiai and slightly modifying the proof of [MR04, Theorem 5.3.10(i)]) applies to conclude with the proof of (ii) as well. The proof of (iii) is identical to the proof of [MR04, Theorem 5.3.10(iii)] (After replacing the Selmer structure denoted by $\mathcal{F}_\Lambda$ on the Λ_{cyc}-adic Galois representation $\mathbf{T}$ in loc.cit. with the Selmer structure $\mathcal{F}_L$ on $X\otimes\Lambda_{\mathrm{cyc}}$). □

Appendix B. $\mathbb{L}$-restricted Kolyvagin systems revisited

In this Appendix we recall a result that the first author proved in [Büy13b] which shows the existence of Kolyvagin systems for the Selmer structure $\mathcal{F}_{\mathbb{L}}$ on $\mathbb{T}$. Even though these Kolyvagin systems do exist unconditionally, they are related (via Theorem A.11) to the $\mathbb{L}$-restricted Kolyvagin system $\boldsymbol{\kappa}^{\Xi}$ obtained from the (conjectural) Perrin-Riou-Stark elements and that we utilized above.

Let $\mathcal{P}$ be the set of places of F that does not contain the archimedean places, primes at which T is ramified and primes above p. Set $r=1+g+\delta$ (where δ is Leopoldt's defect for F) so that $\Gamma\cong\mathbb{Z}_p^r$. Choose a decomposition

$$\Gamma=\Gamma_1\times\cdots\times\Gamma_r$$

where each group Γ_i is isomorphic to $\mathbb{Z}_p$. Fix a topological generator γ_i of the group Γ_i. We then have a (non-canonical) isomorphism

$$\Lambda\cong\mathfrak{O}[[\gamma_1-1,\cdots,\gamma_r-1]]\ .$$

To ease notation, set $R=\Lambda$.

Definition B.1. For $k\in\mathbb{Z}^+$ and $\bar\alpha=(\alpha_1,\cdots,\alpha_r)\in(\mathbb{Z}^+)^r$, set

$$R_{k,\bar\alpha}:=R/(\varpi^k,(\gamma_1-1)^{\alpha_1},\cdots,(\gamma_r-1)^{\alpha_r}),$$
$$\mathbb{T}_{k,\bar\alpha}:=\mathbb{T}\otimes_R R_{k,\bar\alpha}=\mathbb{T}/(\varpi^k,(\gamma_1-1)^{\alpha_1},\cdots,(\gamma_r-1)^{\alpha_r})$$

and define the collection

$$\mathrm{Quot}(\mathbb{T}):=\{\mathbb{T}_{k,\bar\alpha}:k\in\mathbb{Z}^+\text{ and }\bar\alpha=(\alpha_1,\cdots,\alpha_r)\in(\mathbb{Z}^+)^r\}.$$

The propagation of the Selmer structure $\mathcal{F}_{\mathbb{L}}$ (in the sense of [MR04]) to the quotients $\mathbb{T}_{k,\bar\alpha}$ will still be denoted by the symbol $\mathcal{F}_{\mathbb{L}}$ as well as its propagation to T.

Definition B.2. For $\bar{\alpha} \in (\mathbb{Z}^+)^d$ and $k \in \mathbb{Z}^+$, define

(i) $H_{k,\bar{\alpha}} = \ker\big(G_F \to \mathrm{Aut}(\mathbb{T}_{k,\bar{\alpha}}) \oplus \mathrm{Aut}(\boldsymbol{\mu}_{p^k})\big)$,
(ii) $L_{k,\bar{\alpha}} = \overline{F}^{H_{k,\bar{\alpha}}}$,
(iii) $\mathcal{P}_{k,\bar{\alpha}} = \{\text{Primes } \lambda \in \mathcal{P} : \lambda \text{ splits completely in } L_{k,\bar{\alpha}}/F\}$.

The collection $\mathcal{P}_{k,\bar{\alpha}}$ is called the collection of *Kolyvagin primes* for $\mathbb{T}_{k,\bar{\alpha}}$. Define $\mathcal{N}_{k,\bar{\alpha}}$ to be the set of square free products of primes in $\mathcal{P}_{k,\bar{\alpha}}$.

Theorem B.3. *Let $\mathcal{P}_{1,\bar{1}} \subset \mathcal{P}$ be as in Definition B.2.*

(i) *The R-module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}}, \mathcal{P})$ is free of rank one, generated by* any *Kolyvagin system $\boldsymbol{\kappa}$ whose image $\bar{\boldsymbol{\kappa}} \in \mathbf{KS}(\overline{T}, \mathcal{F}_{\mathbb{L}}, \mathcal{P}_{1,\bar{1}})$ is non-zero. Such a Kolyvagin system is called primitive.*
(ii) *Suppose $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}}, \mathcal{P})$ is primitive. Then the image $\boldsymbol{\kappa}^{\mathrm{cyc}}$ of $\boldsymbol{\kappa}$ generates the Λ_{cyc}-module $\overline{\mathbf{KS}}(\mathbb{T}_{\mathrm{cyc}}, \mathcal{F}_{\mathbb{L}}, \mathcal{P})$. Furthermore,*

$$\mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}}(F, \mathbb{T})/\Lambda \cdot \kappa_1\right) \subset \mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(F, \mathbb{T}^*)^\vee\right)$$

and

$$\mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}}(F, \mathbb{T}_{\mathrm{cyc}})/\Lambda \cdot \kappa_1^{\mathrm{cyc}}\right) = \mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(F, \mathbb{T}^*_{\mathrm{cyc}})^\vee\right).$$

Remark B.4. It is the statement of Theorem B.3(ii) that is the key to all our results in the main body of this article. Indeed, we know by Theorem A.11 that the Perrin-Riou-Stark Kolyvagin system $\boldsymbol{\kappa}^{\mathrm{PS}}$ (obtained from Perrin-Riou-Stark elements first applying the twisting morphism tw then the descent procedure in Appendix A with $X = T$) is an element of the cyclic R-module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}}, \mathcal{P})$. We are able to prove (see Theorem 7.7) that this Kolyvagin system is indeed primitive. Since $\kappa_1^{\mathrm{PS}} = \mathrm{tw}(c^{\Xi}_{F_\infty})$ by construction (where we recall that $c^{\Xi}_{F_\infty} \in H^1_{\mathcal{F}_{\mathbb{L}}}(F, \mathbb{T}_\rho)$ is obtained from the conjectural Rubin-Stark elements along the tower F_∞/F following the recipe in Section A.2), the containment in Theorem B.3(ii) translates into

$$\mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}}(F, \mathbb{T})/\Lambda \cdot \mathrm{tw}(c^{\Xi}_{F_\infty})\right) \subset \mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(F, \mathbb{T}^*)^\vee\right) ;$$

and the equality in this theorem into

$$\mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}}(F, \mathbb{T}_{\mathrm{cyc}})/\Lambda \cdot \mathrm{tw}(c^{\Xi}_{F_\infty})^{\mathrm{cyc}}\right) = \mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(F, \mathbb{T}^*_{\mathrm{cyc}})^\vee\right) .$$

We note that all characteristic ideals that appear in these two statements are in fact non-zero, as we have explained in the main body of this text (see Remark 7.5).

The proof of Theorem B.3 is identical to that of [Büy13b, Theorem A.14] and in what follows, we only indicate the key points in the argument and state some of the technical consequences which we also need in the main body of this article.

For $\bar{\alpha} = (\alpha_1, \cdots, \alpha_r), \bar{\beta} = (\beta_1, \cdots, \beta_r) \in (\mathbb{Z}^+)^r$, we write $\bar{\alpha} \prec \bar{\beta}$ (resp., $\bar{\alpha} \succ \bar{\beta}$) whenever $\alpha_i \leq \beta_i$ (resp., whenever $\alpha_i \geq \beta_i$) for all $i = 1, \cdots, r$. The following should be compared to Definition 4.1, Propositions 4.3 and 4.10 of [Büy13a]. This is the key property that allows us to prove Theorem B.3.

Theorem B.5. *Let $\mathcal{F}$ stand for any of the Selmer structures $\mathcal{F}_I$ or $\mathcal{F}_{\mathbb{L}}$ on $\mathbb{T}$. Then $\mathcal{F}$ is cartesian on the collection* Quot$(\mathbb{T}_\rho)$ *in the following sense. Let λ be any prime of F.*

(C1) *For $\bar{\alpha} \prec \bar{\beta}$ and $k \leq k'$, $H^1_{\mathcal{F}}(F_\lambda, \mathbb{T}_{k,\bar{\alpha}})$ is the exact image of $H^1_{\mathcal{F}}(F_\lambda, \mathbb{T}_{k',\bar{\beta}})$ under the canonical map $H^1(F_\lambda, \mathbb{T}_{k',\bar{\beta}}) \to H^1(F_\lambda, \mathbb{T}_{k,\bar{\alpha}})$.*

(C2) *Given $\bar{\alpha} = (\alpha_1, \cdots, \alpha_r)$ as above set $\bar{\alpha}_{+,i} := (\cdots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \cdots)$. Then,*

$$H^1_{\mathcal{F}}(F_\lambda, \mathbb{T}_{k,\bar{\alpha}}) = \ker\left(H^1(F_\lambda, \mathbb{T}_{k,\bar{\alpha}}) \longrightarrow \frac{H^1(F_\lambda, \mathbb{T}_{k,\bar{\alpha}_{+,i}})}{H^1_{\mathcal{F}}(F_\lambda, \mathbb{T}_{k,\bar{\alpha}_{+,i}})}\right).$$

Here the arrow is induced from the injection $\mathbb{T}_{k,\bar{\alpha}} \stackrel{[\gamma_i - 1]}{\longrightarrow} \mathbb{T}_{k,\alpha^-_{+,i}}$ and $[\gamma_i - 1]$ is the multiplication by $\gamma_i - 1$ map.

(C3)

$$H^1_{\mathcal{F}}(F_\lambda, \mathbb{T}_{k,\bar{\alpha}}) = \ker\left(H^1(F_\lambda, \mathbb{T}_{k,\bar{\alpha}}) \stackrel{[\varpi]}{\longrightarrow} \frac{H^1(F_\lambda, \mathbb{T}_{k+1,\bar{\alpha}})}{H^1_{\mathcal{F}}(F_\lambda, \mathbb{T}_{k+1,\bar{\alpha}})}\right),$$

where the arrow is induced from the injection $\mathbb{T}_{k,\bar{\alpha}} \stackrel{[\varpi]}{\longrightarrow} \mathbb{T}_{k+1,\bar{\alpha}}$.

Proof. The proof of this theorem (in even more general form) may be found in our companion article, c.f. the proof of [BL14, Theorem C.8]. The crucial point observed in loc.cit. is that we have a natural identification

$$H^1_{\mathcal{F}}(F_p, \mathbb{T}) \otimes R_{k,\bar{\alpha}} = H^1_{\mathcal{F}}(F_p, \mathbb{T}_{k,\bar{\alpha}})$$

for $\mathcal{F} = \mathcal{F}_I$ or $\mathcal{F}_{\mathbb{L}}$. □

Corollary B.6. *Propagations of both Selmer structures $\mathcal{F}_{\mathbb{L}}$ and $\mathcal{F}_I$ on T verify the hypothesis* **H6** *of* [MR04].

Proposition B.7. *The core Selmer rank $\chi(\overline{T}, \mathcal{F}_{\mathbb{L}})$ equals 1 whereas $\chi(\overline{T}, \mathcal{F}_I)$ equals 0.*

Proof. Both assertions follow from Lemma 6.13. □

Appendix C. Equivariant Coleman maps

Let K be a finite unramified extension of $\mathbb{Q}_p$ containing a completion of F and let E a finite extension of $\mathbb{Q}_p$. We write $H_K = \mathrm{Gal}(\overline{K}/K(\mu_{p^\infty}))$. Throughout this appendix, we fix T a free $\mathcal{O}_E$-module of rank t that is equipped with a continuous action of G_F. Let $r = [K : \mathbb{Q}_p]$, $s = [E : \mathbb{Q}_p]$ and $d = st$. We assume that the hypotheses (H.F.-L.) and (H.S) in [BL14] hold when we regard T as a $\mathbb{Z}_p$-representation of G_F. Only in this appendix $\Lambda = \mathbb{Z}_p[[\Gamma^{\mathrm{cyc}}]]$ and $\Lambda_{\mathcal{O}_E} = \mathcal{O}_E[[\Gamma^{\mathrm{cyc}}]]$.

Lemma C.1. *The Dieudonné module $\mathbb{D}_K(T)$ has a natural $\mathcal{O}_E$-module structure. Moreover, the action of φ on $\mathbb{D}_K(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is E-linear and $\mathbb{D}_K(T)$ is a filtered $\mathcal{O}_E$-module.*

Proof. Recall that $\mathbb{D}_K(T) = \left(\mathbb{A}_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T\right)^{H_K}$. The action of $\mathcal{O}_E$ on T therefore equips $\mathbb{D}_K(T)$ with an $\mathcal{O}_E$-module structure. The action of φ on $\mathbb{D}_K(T)$ is in fact given by the restriction of $\varphi \otimes 1$ on $\mathbb{A}_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T$, hence it commutes with the action of $\mathcal{O}_E$ defined above. Similarly, as $\mathrm{Fil}^i\mathbb{D}_K(T) = (t^i\mathbb{A}_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T)^{H_K}$, it respects the $\mathcal{O}_E$-module structure. □

Let $u_1, \ldots, u_s$ be a $\mathbb{Z}_p$-basis of $\mathcal{O}_E$. Note that $u_i \in \mathcal{O}_E^\times$ for all i.

By Lemma C.1, we may fix an $\mathcal{O}_E$ basis $w_1, \ldots w_{rt}$ of $\mathbb{D}_K(T)$ where $w_1, \ldots, w_{rt_0}$ generate $\mathrm{Fil}^0\mathbb{D}_K(T)$ over $\mathcal{O}_E$. Then $\{u_iw_j : 1 \leq i \leq s, 1 \leq j \leq rt\}$ is a $\mathbb{Z}_p$-basis of $\mathbb{D}_K(T)$.

There is a natural identification

$$\mathcal{H} \otimes_{\mathbb{Z}_p} \mathbb{D}_K(T) \to \mathcal{H}_E \otimes_{\mathcal{O}_E} \mathbb{D}_K(T)$$
$$F \otimes (u_i w_j) \mapsto (u_i F) \otimes w_j.$$

Lemma C.2. *Under the above identification, the regulator map*

$$\mathcal{L}_T^K : H^1_{\mathrm{Iw}}(K,T) \to \mathcal{H}_E \otimes_{\mathcal{O}_E} \mathbb{D}_K(T)$$

is $\Lambda_{\mathcal{O}_E}$-linear.

Proof. If we write $D_K(T) = (\mathbb{B} \otimes_{\mathbb{Z}_p} T)^{H_K}$ for the (φ, Γ)-module of T, we may identify $H^1_{\mathrm{Iw}}(K,T)$ with $D_K(T)^{\psi=1}$. Under this identification, $\mathcal{L}_T^K$ is given by $\mathfrak{M} \circ (1-\varphi)$, where $\mathfrak{M}$ is the Mellin transform (see [LLZ11, Definition 3.4]). Since φ acts on $\mathbb{B}$, but not T, whereas $\mathcal{O}_E$ acts on T, but not $\mathbb{B}$, the two actions commute. Hence, we are done. □

Therefore, we may write

$$\mathcal{L}_T^K = \sum_{i=1}^{rt} w_i \mathcal{L}_{T,i}^K$$

where $\mathcal{L}_{T,i}^K$ are $\Lambda_{\mathcal{O}_E}$-linear maps from $H^1_{\mathrm{Iw}}(K,T)$ to $\mathcal{H}_E$.

By Lemma C.1, the action of φ on $\mathbb{D}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is E-linear. Let C_φ^E be the matrix of φ with respect to the basis $\{w_j\}$. Then, as in [BL14, §2.2], C_φ^E is of the form

$$C \left(\begin{array}{c|c} I_{rt_0} & 0 \\ \hline 0 & \frac{1}{p} I_{r(t-t_0)} \end{array} \right)$$

for some $C \in \mathrm{GL}_d(\mathcal{O}_E)$. This allows us to construct a logarithmic matrix M_T^E whose entries are all $o(\log)$ with determinant, upto a unit in $\mathcal{O}_E$, $\left(\frac{\log(1+X)}{pX}\right)^{r(t-t_0)}$. The same calculations as in [BL14, §2.3] shows that there is a decomposition

$$\mathcal{L}_T^K = \begin{pmatrix} w_1 & \cdots w_{rt} \end{pmatrix} \cdot M_T^E \cdot \mathrm{Col}_T^{K,E}.$$

where $\mathrm{Col}_T^{K,E} : H^1_{\mathrm{Iw}}(K,T) \to \Lambda_{\mathcal{O}_E}^{\oplus rt}$ is $\Lambda_{\mathcal{O}_E}$-linear.

Recall the the reciprocity law of Colmez-Perrin-Riou states that the determinant of $\mathcal{L}_T^K$ over Λ is, upto a unit, equal to $\left(\frac{\log(1+X)}{p}\right)^{r(d-d_0)}$. But

$$\det{}_\Lambda \mathcal{L}_T^K = \left(\det{}_{\Lambda_{\mathcal{O}_E}} \mathcal{L}_T^K\right)^s,$$

so the determinant over $\Lambda_{\mathcal{O}_E}$ is, upto a unit, equal to $\left(\frac{\log(1+X)}{p}\right)^{t(d-d_0)}$. Therefore, we may carry out the same calculations as in [BL14, §2.4] and conclude that for any subset $I \in \{1, \ldots, rt\}$ and any character η of conductor p or 1,

$$\det \left(\mathrm{Im} \left(\bigoplus_{i \in I} \mathrm{Col}_{T,i}^{K,E} \right)^\eta \right) = X^{n(I,\eta)}$$

for some integer $n(I,\eta) \geq 0$. Furthermore, when the basis of $\mathbb{D}_p(T)$ we have fixed is strongly admissible in the sense of [BL14, Definition 3.2] then we may take $n(I,\eta) = 0$.

REFERENCES

[BL14] Kâzım Büyükboduk and Antonio Lei. Integral Iwasawa theory of motives for non-ordinary primes, 2014. in preparation, draft available upon request.

[Büy09] Kâzım Büyükboduk. Stark units and the main conjectures for totally real fields. *Compositio Math.*, 145:1163–1195, 2009.

[Büy10] Kâzim Büyükboduk. On Euler systems of rank r and their Kolyvagin systems. *Indiana Univ. Math. J.*, 59(4):1277–1332, 2010.

[Büy13a] Kâzım Büyükboduk. Deformations of Kolyvagin systems, 2013. Preprint available at `http://arxiv.org/abs/1303.1407`.

[Büy13b] Kâzım Büyükboduk. Main conjectures for CM fields and a Yager-type theorem for Rubin-Stark elements, 2013. *IMRN*, to appear.

[Büy13c] Kâzım Büyükboduk. On the Iwasawa theory of CM fields for supersingular primes, 2013. submitted.

[Hid06] Haruzo Hida. Anticyclotomic main conjectures. *Doc. Math.*, (Extra Vol.):465–532 (electronic), 2006.

[Hid09] Haruzo Hida. Quadratic exercises in Iwasawa theory. *Int. Math. Res. Not. IMRN*, (5):912–952, 2009.

[Hsi14] Ming-Lun Hsieh. Eisenstein congruence on unitary groups and Iwasawa main conjectures for CM fields. *J. Amer. Math. Soc.*, 27(3):753–862, 2014.

[Kob03] Shin-ichi Kobayashi. Iwasawa theory for elliptic curves at supersingular primes. *Invent. Math.*, 152(1):1–36, 2003.

[LLZ11] Antonio Lei, David Loeffler, and Sarah Livia Zerbes. Coleman maps and the p-adic regulator. *Algebra Number Theory*, 5(8):1095–1131, 2011.

[Mai08] Fabio Mainardi. On the main conjecture for CM fields. *Amer. J. Math.*, 130(2):499–538, 2008.

[MR04] Barry Mazur and Karl Rubin. Kolyvagin systems. *Mem. Amer. Math. Soc.*, 168(799):viii+96, 2004.

[Nek06] Jan Nekovář. Selmer complexes. *Astérisque*, (310):viii+559, 2006.

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2008.

[Och05] Tadashi Ochiai. Euler system for Galois deformations. *Ann. Inst. Fourier (Grenoble)*, 55(1):113–146, 2005.

[Pol03] Robert Pollack. On the p-adic L-function of a modular form at a supersingular prime. *Duke Math. J.*, 118(3):523–558, 2003.

[PR84] Bernadette Perrin-Riou. Arithmétique des courbes elliptiques et théorie d'Iwasawa. *Mém. Soc. Math. France (N.S.)*, (17):130, 1984.

[PR98] Bernadette Perrin-Riou. Systèmes d'Euler p-adiques et théorie d'Iwasawa. *Ann. Inst. Fourier (Grenoble)*, 48(5):1231–1307, 1998.

[PR04] Robert Pollack and Karl Rubin. The main conjecture for CM elliptic curves at supersingular primes. *Ann. of Math. (2)*, 159(1):447–464, 2004.

[Rib76] Kenneth A. Ribet. Dividing rational points on Abelian varieties of CM-type. *Compos. Math.*, 33:69–74, 1976.

[Rub91] Karl Rubin. The "main conjectures" of Iwasawa theory for imaginary quadratic fields. *Invent. Math.*, 103(1):25–68, 1991.

[Rub96] Karl Rubin. A Stark conjecture "over $\mathbf{Z}$" for abelian L-functions with multiple zeros. *Ann. Inst. Fourier (Grenoble)*, 46(1):33–62, 1996.

[Rub00] Karl Rubin. *Euler systems*, volume 147 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study.

[ST68] Jean-Pierre Serre and John Tate. Good reduction of abelian varieties. *Ann. of Math. (2)*, 88:492–517, 1968.

[Sug12] Ken-ichi Sugiyama. On a generalization of Deuring's results. arXiv: 1212.5883, 2012.

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

Kâzim Büyükboduk
Koç University, Mathematics, Rumeli Feneri Yolu, 34450 Sariyer, Istanbul, Turkey
E-mail address: kbuyukboduk@ku.edu.tr

Antonio Lei
Département de Mathématiques et de Statistique, Université Laval, Pavillion Alexandre-Vachon, 1045 Avenue de la Médecine, Québec, QC, Canada G1V 0A6
E-mail address: antonio.lei@mat.ulaval.ca