Congruences modulo p between ρ-twisted Hasse-Weil L-values

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Abstract: *Suppose E_1 and E_2 are semistable elliptic curves over $\mathbb{Q}$ with good reduction at p, whose associated weight two newforms f_1 and f_2 have congruent Fourier coefficients modulo p. Let $R_S(E_\star, \rho)$ denote the algebraic p-adic L-value attached to each elliptic curve $E_\star$, twisted by an irreducible Artin representation, ρ, factoring through the Kummer extension $\mathbb{Q}\big(\mu_{p^\infty}, \Delta^{1/p^\infty}\big)$.*

If E_1 and E_2 have good ordinary reduction at p, we prove that

$$R_S(E_1, \rho) \equiv R_S(E_2, \rho) \mod p,$$

under an integrality hypothesis for the modular symbols defined over the field cut out by $\mathrm{Ker}(\rho)$. *Under this hypothesis, we establish that E_1 and E_2 have the same analytic λ-invariant at ρ.*

Alternatively, if E_1 and E_2 have good supersingular reduction at p, we show that

$$\big|R_S(E_1, \rho) - R_S(E_2, \rho)\big|_p \ < \ p^{\mathrm{ord}_p(\mathrm{cond}(\rho))/2}.$$

These congruences generalise some earlier work of Vatsal [Vat99], Shekhar-Sujatha [SS15] and Choi-Kim [CK16], to the false Tate curve setting.

Contents

1 Introduction

Let p be a prime. Suppose $\mathcal{F} = \sum t_m(\mathcal{F})q^m$ and $\mathcal{G} = \sum t_m(\mathcal{G})q^m$ are two modular cusp forms, such that their Fourier coefficients satisfy $t_m(\mathcal{F}) \equiv t_m(\mathcal{G}) \pmod{p^r}$ at every integer $m \geq 1$. Vatsal showed in his seminal article [Vat99] that, under an appropriate choice of complex period, the algebraic parts of the L-functions for $\mathcal{F}$ and $\mathcal{G}$ twisted by a Dirichlet character χ, will be congruent modulo p^r. Moreover, if p is a good ordinary prime for $\mathcal{F}$ and $\mathcal{G}$, his result directly implies that their p-adic L-functions must also be congruent mod p^r.

Vatsal's theorem has many applications. For example, using this congruence one may relate the Iwasawa invariants of eigenforms inside a Hida deformation, thereby facilitating the study of Iwasawa main conjectures over ordinary families (see [GV00] and [EPW06]). It is natural to ask: Can Vatsal's result be generalized to twists by more exotic representations?

In [SS14, SS15], Shekhar and Sujatha studied precisely this question for dihedral twists in the weight two case, under the assumption that the cusp forms $\mathcal{F}$ and $\mathcal{G}$ are ordinary at p. Using the same p-ordinarity assumption, Kriz and Li [KL16] established congruences between Heegner points associated to congruent elliptic curves, which resulted in a mod p relation for the anticyclotomic p-adic L-function constructed by Bertolini-Darmon-Prasanna in [BDP13]. This enabled them to obtain information on both the Birch-Swinnerton-Dyer Conjecture, and the Goldfeld Conjecture.

For the supersingular case, Kim [Kim14] has proven that there exists a p-adic L-function which interpolates the complex L-values of $\mathcal{F}$ and $\mathcal{G}$, twisted by Hecke characters over an imaginary quadratic field in which p splits. Whilst these elements are not p-integral, their denominators are controlled in an explicit manner. In [CK16], Choi and Kim showed under certain technical conditions similar to those in [Vat99], that these non-integral p-adic L-functions satisfy congruence relations (in the sense that, when evaluated at the appropriate characters, the p-adic norm of their difference is smaller than that of their original norms).

Goal. *The principal aim of the present article is to establish a modulo p congruence relation for the Hasse-Weil L-functions of congruent elliptic curves, twisted by an Artin representation 'ρ' factoring through a Kummer extension of $\mathbb{Q}$.*

We shall treat both the p-ordinary and the p-supersingular cases here.

Let E_1 and E_2 denote elliptic curves defined over $\mathbb{Q}$, with conductors N_1 and N_2 respectively. In particular both E_1 and E_2 are modular by the work of Wiles et al, which means there are associated primitive forms $f_\star \in \mathcal{S}_2^{\text{new}}(\Gamma_0(N_\star))$ satisfying $L(f_\star, s) = L(E_\star, s)$ for $\star = 1, 2$. Henceforth we fix a prime number $p \neq 2$, and impose the following conditions:

(i) both curves E_1 and E_2 have good reduction at the prime p;

(ii) their conductors N_1 and N_2 are square-free integers;

(iii) the Fourier coefficients $t_m(E_1) \equiv t_m(E_2) \pmod{p}$ whenever $\gcd(m, N_1N_2) = 1$.

The condition (ii) is necessary as our expressions for the Rankin convolution are only valid provided the two curves E_1 and E_2 are semistable over the ground field. Also (iii) is the same condition as in [SS15], and weaker than insisting $t_m(E_1) \equiv t_m(E_2) \pmod{p}$ for every $m \geq 1$.

In order to obtain algebraic L-values, we first need to divide our L-functions by the correct motivic periods, and then hope that the special values display some kind of integral structure. For the semistable case these integrality issues were first raised in Bouganis's thesis [Bou06], while in the case of elliptic curves with complex multiplication, there is a detailed analysis of period ratios in [DW10]. We shall adopt the same period choices made by Vatsal in [Vat99].

If $M=\mathrm{lcm}(N_1,N_2)$, then both f_1 and f_2 are eigenforms on the congruence subgroup

$$\Gamma_1(M) \ := \ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ where } a\equiv d\equiv 1 \ (\bmod M) \text{ and } c\equiv 0 \ (\bmod M) \right\}.$$

Let $\mathbf{T}$ be the Hecke algebra acting on the set of cusp forms of weight 2 on the subgroup $\Gamma_1(M)$. The congruence class of f_1 and f_2 determines a maximal ideal $\mathfrak{m}$ of $\mathbf{T}$, and for $\star\in\{1,2\}$ each eigenform $f_\star$ will then induce a $\mathbb{Q}_p$-algebra homomorphism

$$\pi_\star : \mathbf{T}_{\mathfrak{m}}\otimes\mathbb{Q}_p \to \mathbb{Q}_p.$$

We write $R_{f_\star}$ for the unique local factor of the Hecke algebra through which $\pi_\star$ factors.

Hypothesis (Vat). *The local factors satisfy $R_{f_1}=R_{f_2}=\mathbb{Q}_p$, and there exist $\mathbf{T}$-isomorphisms*

$$H^1\big(\Gamma_1(M),\mathbb{Z}_p\big)^{\pm}_{\mathfrak{m}} \ = \ H^1_P\big(\Gamma_1(M),\mathbb{Z}_p\big)^{\pm}_{\mathfrak{m}} \ \cong \ \mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{T},\mathbb{Z}_p)_{\mathfrak{m}} \ = \ S_2\big(\Gamma_1(M);\mathbb{Z}_p\big)_{\mathfrak{m}},$$

where $H^1_P\big(\Gamma_1(M),\mathbb{Z}_p\big)$ is the parabolic subgroup of $H^1\big(\Gamma_1(M),\mathbb{Z}_p\big)$, and $\pm$ denote the eigenspaces for the complex conjugation.

For example, the equality $R_{f_1}=R_{f_2}=\mathbb{Q}_p$ is certainly true if both elliptic curves E_1 and E_2 share the same conductor, since both f_1 and f_2 are then newforms of level $M=N_1=N_2$. The second freeness hypothesis is discussed at length in [Vat99, §2]. As explained in *loc. cit.*, this hypothesis produces canonical periods, $\Omega^{\pm}_{f_\star}$, which are well-defined up to p-adic units. Furthermore in the case of elliptic curves, these periods in fact coincide with the standard Néron periods.

For a positive integer n we set $K_n=\mathbb{Q}(\mu_{p^n})$, and write $F_n=\mathbb{Q}(\mu_{p^n})^+$ for its maximal totally real subfield. Let $\mathfrak{p}$ be the unique place of F_n lying above p, and S will denote a finite set of places for F_n containing $\mathfrak{p}$. We label by $\alpha_\star(p)$ and $\beta_\star(p)$ the two roots of the Hecke polynomial

$$P_p(E_\star\,,X) \ = \ X^2-t_p(E_\star)X+p.$$

Throughout we normalise the p-adic valuation $\big|-\big|_p$ on the Tate field $\mathbb{C}_p=\widehat{\overline{\mathbb{Q}}}_p$ by $\big|p\big|_p=p^{-1}$. Let us also fix embeddings $\iota_\infty:\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$ and $\iota_p:\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}_p$. These embeddings will allow us to consider the field of algebraic numbers, $\overline{\mathbb{Q}}$, simultaneously as both complex and p-adic numbers.

For the rest of the Introduction, we assume S is a finite set of places of the number field F_n containing always those places dividing $p\cdot\Delta\cdot N_1\cdot N_2$.

Definition 1.1. *For an Artin representation $\rho:\mathrm{Gal}(\overline{\mathbb{Q}}/F_n)\to\mathrm{GL}(V)$ and $\star\in\{1,2\}$, we define*

$$R_S(E_\star\,,\rho) \ := \ \iota_\infty^{-1}\left(\frac{\epsilon_{F_n}(\rho^*)}{\alpha_\star(p)^{f(\rho,\mathfrak{p})}}\times\frac{P_{\mathfrak{p}}\big(\rho,\alpha_\star(p)^{-1}\big)}{P_{\mathfrak{p}}\big(\rho^*,\beta_\star(p)^{-1}\big)}\times\frac{L_S(E_\star\,,\rho^*,1)}{\big(2\pi i\Omega^+_{f_\star}\big)^{\dim(\rho^+)}\big(2\pi i\Omega^-_{f_\star}\big)^{\dim(\rho^-)}}\right)$$

where ρ^* means the contragredient representation to ρ, the notation $\epsilon_{F_n}(\rho^*)$ indicates the local epsilon factor for ρ^* at p, the power $f(\rho,\mathfrak{p})$ represents the $\mathfrak{p}$-exponent of the conductor of ρ, while $P_{\mathfrak{p}}(\rho,X)$ and $P_{\mathfrak{p}}(\rho^*,X)$ denote the characteristic polynomials (associated with ρ and ρ^*, respectively) at p.

We call $R_S(E_\star\,,\rho)$ the *algebraic part* of the L-value of $E_\star$, twisted by the representation ρ^*.

Let $\Delta>1$ be a p-power free integer, and assume $\chi:\mathrm{Gal}\big(K_n(\Delta^{1/p^n})/K_n\big)\twoheadrightarrow\mu_{p^n}$ is the surjective character sending $\sigma\mapsto\frac{\sigma(\Delta^{1/p^n})}{\Delta^{1/p^n}}$. We now consider the irreducible representation

$$\rho \ = \ \psi\otimes\mathrm{Ind}^{F_n}_{K_n}(\chi)$$

where ψ is a character on $\mathrm{Gal}(F_{n+m}/F_n)$ for some integer $m \geq 0$. In Section 5 we shall introduce a technical condition, Hypothesis $(\mathrm{MS})_p$, which is unfortunately too long to give here. Essentially, the Hypothesis $(\mathrm{MS})_p$ ensures that for all ψ-twists as above satisfy

$$\iota_p\left(R_S\left(E_\star\,, \psi \otimes \mathrm{Ind}_{K_n}^{F_n}(\chi)\right)\right) \;\in\; \mathcal{O}_{\mathbb{C}_p}.$$

In the p-ordinary case, these values are interpolated by a power series $\mathbf{L}_p\big(E_\star\,, \mathrm{Ind}(\chi)\big) \in \mathbb{Q}_p[\![X]\!]$ (see [Bou06, DW08]); thus $(\mathrm{MS})_p$ is equivalent to demanding that $\mathbf{L}_p\big(E_\star\,, \mathrm{Ind}(\chi)\big) \in \mathbb{Z}_p[\![X]\!]$. In this situation, the χ-twisted Iwasawa Main Conjecture implies that Hypothesis $(\mathrm{MS})_p$ must automatically be true as well, since $\mathbf{L}_p\big(E_\star\,, \mathrm{Ind}(\chi)\big)$ would then be a characteristic power series for the χ-twisted Selmer group, which is always p-integral.

Theorem 1.2. *If E_1 and E_2 have good ordinary reduction at p, and Hypothesis $(MS)_p$ holds:*

(i) $R_S(E_1, \rho)$ and $R_S(E_2, \rho)$ are both p-integral;

(ii) $R_S(E_1, \rho) \equiv R_S(E_2, \rho) \mod p$.

Moreover if Hypothesis (Vat) holds true as well, then (ii) gives a ***non-trivial*** *congruence for the representation $\rho = \psi \otimes \mathrm{Ind}_{K_n}^{F_n}(\chi)$ at infinitely many ψ.*

We should point out that Shekhar and Sujatha have proved an algebraic analogue of this result, which concerns Euler characteristics of the fine Selmer groups in [SS14]. They also obtain in [SS15] a similar result to our modulo p congruence, but for dihedral representations instead; the two results coincide precisely when $p = 3$.

Note that in the supersingular case, the L-value $R_S(E_\star, \rho)$ is no longer p-integral due to the presence of the term $\alpha_\star(p)$ in the denominator, the obstruction being that $\big|\alpha_\star(p)\big|_p = p^{-1/2}$. Nevertheless, one can always consider the p-adic norm of $R_S(E_1, \rho) - R_S(E_2, \rho)$ as a substitute, which is the approach taken by Choi and Kim in [CK16].

Theorem 1.3. *If E_1 and E_2 have supersingular reduction at p, and Hypothesis $(MS)_p$ holds:*

(i) $\big|R_S(E_1, \rho)\big|_p \;\leq\; p^{f(\rho,\mathfrak{p})/2}$ and $\big|R_S(E_2, \rho)\big|_p \;\leq\; p^{f(\rho,\mathfrak{p})/2}$;

(ii) Moreover, if $\big|\alpha_1(p) - \alpha_2(p)\big|_p < p^{-1/2}$, then

$$\Big|R_S(E_1, \rho) - R_S(E_2, \rho)\Big|_p \;<\; p^{f(\rho,\mathfrak{p})/2}\,.$$

One observes that Weil's bound tells us that $t_p(E_i) = 0$ if $p \geq 5$, and $t_p(E_i) \in \{0, \pm 3\}$ if $p = 3$. If $t_p(E_1) = t_p(E_2) = 0$, then we may always choose $\alpha_1(p) = \alpha_2(p) = \pm\sqrt{-p}$. Alternatively, if both $t_p(E_1)$ and $t_p(E_2)$ are non-zero then $\alpha_\star(p) = \frac{\pm 3 \pm \sqrt{-3}}{2}$, in which case there always exists a pair $\alpha_1(p)$, $\alpha_2(p)$ satisfying the condition (ii). However, if exactly one of $t_p(E_1)$ and $t_p(E_2)$ vanishes (which can only happens when $p = 3$), then clearly such a pair cannot exist.

We note that the context of [CK16] is actually quite different from ours. In *loc. cit.* the authors studied congruences between two modular forms of *different* weights, and χ is taken to be a Hecke character over an imaginary quadratic field where p splits. Furthermore, in their setting the values $R_S(E_\star, \rho)$ in fact deform continuously into a two-variable p-adic L-function. In our specific situation, it is not really clear to us whether such an object could feasibly exist. However it is still striking to us that a similar congruence relation holds, even without the existence of such a p-adic L-function. While we have concentrated on congruent elliptic curves, the majority of the calculations can be generalized to modular forms of arbitrary weight $k \geq 2$ – see [War12] for some results in this direction.

The strategy behind our proof of Theorems 1.2 and 1.3 is very similar to that in [SS15]. The first step is to show that for $\star \in \{1,2\}$, there is a p-adic congruence between $R_S(E_\star,\rho)$ and $R_S\big(E_\star,\psi\otimes \mathrm{Ind}_{K_n}^{F_n}\mathbf{1}\big)$. This is achieved by rewriting $R_S(E_\star,-)$ in terms of the Petersson inner product of two Hilbert modular forms. By Atkin-Lehner theory, these algebraic L-values decompose into a finite sum involving the coefficients in the q-expansion, which then allows us to read off the aforementioned congruence from the respective Fourier expansions. (We note that such calculations were partially carried out in [DW08], albeit only in the ordinary case.)

The second step is to compare $R_S\big(E_1,\psi\otimes \mathrm{Ind}_{K_n}^{F_n}\mathbf{1}\big)$ with $R_S\big(E_2,\psi\otimes \mathrm{Ind}_{K_n}^{F_n}\mathbf{1}\big)$ p-adically. Because the representation $\mathrm{Ind}_{K_n}^{\mathbb{Q}}(\mathbf{1})$ splits completely into a direct sum of Dirichlet characters, the Artin formalism implies that both $R_S\big(E_1,\psi\otimes \mathrm{Ind}_{K_n}^{F_n}\mathbf{1}\big)$ and $R_S\big(E_2,\psi\otimes \mathrm{Ind}_{K_n}^{F_n}\mathbf{1}\big)$ have a product decomposition, over the same set of Dirichlet twists. The various fragments can then be shown to be congruent modulo p individually, upon applying the main results in [Vat99]. Combining the congruences from Steps 1 and 2 together yields the two theorems above.

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2 Background on Hilbert modular forms

Our main references for this preliminary material on HMFs are the articles [Shi78] and [Pan91]. Let F be a totally real field of degree $d=[F:\mathbb{Q}]$, and write $\mathfrak{d}=\mathfrak{d}_F$ for its different ideal. One may then interpret $\mathrm{GL}_2(F)$ as a group $\mathfrak{G}_{\mathbb{Q}}$ of rational points for an associated $\mathbb{Q}$-subgroup inside $\mathrm{GL}_{2d}(\mathbb{Q})$. Its adèlisation $\mathfrak{G}_{\mathbb{A}}$ corresponds to the product

$$\mathrm{GL}_2(\mathbb{A}_F) \;=\; \mathrm{GL}_2(\mathbb{R})^d\times \mathrm{GL}_2(\hat{F})\quad\text{where}\;\; \hat{F}:=F\otimes\big(\varprojlim_m \mathbb{Z}/m\mathbb{Z}\big).$$

The proper subgroup $\mathrm{GL}_2^+(\mathbb{R})^d$ comprising vectors $\underline{v}=\big(v_1,\dots,v_d\big)$ with $v_j=\begin{pmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{pmatrix}$ and $\alpha_j\delta_j>\beta_j\gamma_j$ for all $j\le d$, acts naturally on d-copies of the upper half-plane $\mathfrak{H}$. If $\mathbf{i}=(i,\dots,i)$, there is an isomorphism $\big\{\underline{v}\in\mathrm{GL}_2^+(\mathbb{R})^d\;\big|\;\underline{v}\cdot\mathbf{i}=\mathbf{i}\big\}/\mathbb{R}_+^d\cong \mathrm{SO}(2)^d$ and this quotient is maximally compact within $\mathrm{GL}_2(\mathbb{R})^d/\mathbb{R}_+^d$.

Remarks: (a) For any element $\underline{v}\in\mathrm{GL}_2^+(\mathbb{R})^d$ and function $f:\mathfrak{H}^d\to\mathbb{C}$,

$$\big(f\big|_k\underline{v}\big)(z) \;:=\; \mathcal{N}\big(\gamma_j z_j+\delta_j\big)^{-k}\times f\left(\underline{v}\cdot z\right)\cdot\mathcal{N}\big(\det(\underline{v})\big)^{k/2}\;\text{ at integers } k>0$$

where the norm of an d-tuple $z=(z_1,\dots,z_d)$ is given by $\mathcal{N}(z)=z_1\times\cdots\times z_d$.

(b) Let $\mathfrak{c}$ be an ideal of $\mathcal{O}_F$; one has localisations $\mathfrak{c}_{\mathfrak{q}}=\mathfrak{c}\cdot\mathcal{O}_{F,\mathfrak{q}}$ and $\mathfrak{d}_{\mathfrak{q}}=\mathfrak{d}\cdot\mathcal{O}_{F,\mathfrak{q}}$. We define open subgroups $W_{\mathfrak{c}}\subset\mathfrak{G}_{\mathbb{A}}$ by the product $W_{\mathfrak{c}}:=\mathrm{GL}_2^+(\mathbb{R})^d\times\prod_{\mathfrak{q}}W(\mathfrak{q})$, with each local factor consisting of matrices

$$W(\mathfrak{q})=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(F_{\mathfrak{q}})\;\middle|\; b\in\mathfrak{d}_{\mathfrak{q}}^{-1},\; c\in\mathfrak{d}_{\mathfrak{q}}\mathfrak{c}_{\mathfrak{q}},\;\; a,d\in\mathcal{O}_{F,\mathfrak{q}},\;\; ad-bc\in\mathcal{O}_{F,\mathfrak{q}}^{\times}\right\}.$$

(c) If $\hat{h}_F=\#\mathrm{Cl}^{\mathrm{nw}}(\mathcal{O}_F)$ denotes the narrow class number of F, one can always choose ideles $t_1,\dots,t_{\hat{h}_F}\in\mathbb{A}_F^\times$ so that their associated $\mathcal{O}_F$-ideals $\widetilde{t}_1,\dots,\widetilde{t}_{\hat{h}_F}$ form a complete set of representatives for $\mathrm{Cl}^{\mathrm{nw}}(\mathcal{O}_F)$. By the approximation theorem

$$\mathfrak{G}_{\mathbb{A}} \;=\; \bigcup_\lambda \mathfrak{G}_{\mathbb{Q}}\cdot x_\lambda\cdot W_{\mathfrak{c}} \;=\; \bigcup_\lambda \mathfrak{G}_{\mathbb{Q}}\cdot\big(x_\lambda^{-1}\big)^{\iota}\cdot W_{\mathfrak{c}}$$

where the elements $x_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & t_\lambda \end{pmatrix}$, and the involution $\iota : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Definition 2.1. *(a) Fix a weight $k > 0$, an ideal $\mathfrak{c}$, and a Hecke character ψ mod $\mathfrak{c}$. Then a Hilbert automorphic form $\mathbf{f} : \mathfrak{G}_{\mathbb{A}} \to \mathbb{C}$ of parallel weight $(k, \ldots, k)$, level $\mathfrak{c}$ and character ψ satisfies:*

(i) $\mathbf{f}(sgx) = \psi(s) \cdot \mathbf{f}(x)$ *for all* $x \in \mathfrak{G}_{\mathbb{A}}$, $s \in \mathbb{A}_F^\times$ *and* $g \in \mathfrak{G}_{\mathbb{Q}}$;

(ii) $\mathbf{f}(xw) = \psi(w^\iota) \cdot \mathbf{f}(x)$ *for every* $w \in W_{\mathfrak{c}}$ *with* $w_\infty = 1$;

(iii) $\mathbf{f}\big(xr(\theta)\big) = \mathbf{f}(x) \cdot \exp\big(ik\{\theta\}\big)$ *where* $r(\theta) = \left(\ldots, \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix}, \ldots\right)$.

(b) An automorphic form $\mathbf{f} : \mathfrak{G}_{\mathbb{A}} \to \mathbb{C}$ is cuspidal provided that

$$\int_{\mathbb{A}_F/F} \mathbf{f}\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot g\right) \cdot \mathrm{d}t \;=\; 0 \quad \text{at each element } g \in \mathfrak{G}_{\mathbb{A}}.$$

If $\mathbf{f}$ satisfies the condition that for any $x \in \mathfrak{G}_{\mathbb{A}}$ with archimedean component $x_\infty = 1$ there exists $h_x : \mathfrak{H}^d \to \mathbb{C}$ such that $\mathbf{f}(xy) = \big(h_x\big|_k \underline{v}\big)(\mathbf{i})$ for all vectors $\underline{v} \in \mathrm{GL}_2^+(\mathbb{R})^n$ with each h_x holomorphic at the cusps, then $\mathbf{f} : \mathfrak{G}_{\mathbb{A}} \to \mathbb{C}$ is a Hilbert modular form of holomorphic type.

From Remark (c) above, these HMFs correspond to $\hat{h}_F$-tuples $(f_1, \ldots f_{\hat{h}_F})$ of functions on $\mathfrak{H}^d$. If $\mathfrak{c}$ is an ideal of F, then we write $\mathcal{M}_k(\mathfrak{c}, \psi)$ for the space of Hilbert automorphic forms of parallel weight k, level $\mathfrak{c}$, with finite order character ψ. Specifically, if $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ then

$$f_\lambda\big|_k \gamma \;=\; \psi(\gamma)\, f_\lambda$$

for all $\lambda = 1, \ldots, \hat{h}_F$ and $\gamma \in \Gamma_\lambda(\mathfrak{c})$, where $\Gamma_\lambda(\mathfrak{c})$ is the congruence modular group is given by

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;:\; b \in \tilde{t}_\lambda^{-1}\mathfrak{d}_F^{-1},\; c \in \tilde{t}_\lambda \mathfrak{c}\, \mathfrak{d}_F,\; a, d \in \mathcal{O}_F,\; ad - bc \in \mathcal{O}_F^\times \right\}.$$

One then defines

$$e_F(\xi z) = \exp\left(2\pi i \sum_{1 \le a \le d} \xi^{\tau_a} z_a\right)$$

for $z = (z_1, \ldots, z_d) \in \mathfrak{H}^d$, $\xi \in F$, and where $\tau_1, \ldots, \tau_d : F \hookrightarrow \mathbb{R}$ are distinct embeddings.

Properties: (i) Each individual component f_λ has a Fourier expansion

$$f_\lambda(z) = \sum_\xi a_\lambda(\xi)\, e_F(\xi z),$$

and the summation is taken over totally positive elements $\xi \in \tilde{t}_\lambda$ and $\xi = 0$.

(ii) For a subring $R \subset \mathbb{C}$, we use $\mathcal{M}_k(\mathfrak{c}, \psi; R)$ to denote the R-submodule of forms $\mathbf{f}$ in $\mathcal{M}_k(\mathfrak{c}, \psi)$, whose Fourier coefficients $a_\lambda(\xi)$ belong to R for every $\xi \gg 0$ in $\tilde{t}_\lambda$, and for $\xi = 0$.

(iii) If $\mathbf{f}$ is itself a cusp form, then the constant terms $a_\lambda(0) = 0$ for all λ; the vector subspace of cusp forms of parallel weight k, level $\mathfrak{c}$ and character ψ is written as $\mathcal{S}_k(\mathfrak{c}, \psi)$.

(iv) A cusp form $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ possesses Fourier coefficients $C(\mathfrak{m}, \mathbf{f})$, given by

$$C(\mathfrak{m}, \mathbf{f}) \;=\; \begin{cases} a_\lambda(\xi) N_{F/\mathbb{Q}}(\tilde{t}_\lambda)^{-k/2} & \text{if the ideal } \mathfrak{m} = \xi \tilde{t}_\lambda^{-1} \text{ is integral} \\ 0 & \text{if } \mathfrak{m} \text{ is not integral.} \end{cases}$$

We will employ some standard linear operators on the spaces $\mathcal{M}_k(\mathfrak{c},\psi)$ and $\mathcal{S}_k(\mathfrak{c},\psi)$.

Let $\mathfrak{q}$ be an integral ideal of the ring $\mathcal{O}_F$, and q an idèle such that $\tilde{q}=\mathfrak{q}$. One defines the operators $\mathfrak{q}$ and $U(\mathfrak{q})$ on $\mathbf{f}\in\mathcal{M}_k(\mathfrak{c},\psi)$ by

$$\begin{aligned} (\mathbf{f}|\mathfrak{q})(x) &= N_{F/\mathbb{Q}}(\mathfrak{q})^{-k/2}\,\mathbf{f}\left(x\begin{pmatrix} q & 0\\ 0 & 1\end{pmatrix}\right) \\ \text{and}\quad \big(\mathbf{f}|U(\mathfrak{q})\big)(x) &= N_{F/\mathbb{Q}}(\mathfrak{q})^{k/2-1}\sum_{v\in\mathcal{O}_F/\mathfrak{q}}\mathbf{f}\left(x\begin{pmatrix} 1 & v\\ 0 & q\end{pmatrix}\right). \end{aligned}$$

These operators can be described by their effects on the Fourier coefficients, namely

$$C(\mathfrak{m},\mathbf{f}\big|\mathfrak{q}) = C(\mathfrak{m}\mathfrak{q}^{-1},\mathbf{f}) \qquad\text{and}\qquad C\big(\mathfrak{m},\mathbf{f}|U(\mathfrak{q})\big) = C(\mathfrak{m}\mathfrak{q},\mathbf{f}).$$

We also require the operator $J_{\mathfrak{c}}$ (which is an involution at even weight k), and is defined by

$$(\mathbf{f}|J_{\mathfrak{c}})(x) = \psi(\det(x)^{-1})\,\mathbf{f}\left(x\begin{pmatrix} 0 & 1\\ c_0 & 0\end{pmatrix}\right)$$

where c_0 is an idèle such that $\tilde{c}_0=\mathfrak{c}\mathfrak{d}_F^2$; in particular, $\mathbf{f}|J_{\mathfrak{c}}\in\mathcal{M}_k(\mathfrak{c},\psi^{-1})$. This mapping has the additional property

$$\mathbf{f}|J_{\mathfrak{m}\mathfrak{c}} = N_{F/\mathbb{Q}}(\mathfrak{m})^{k/2}(\mathbf{f}|J_{\mathfrak{c}})\big|\mathfrak{m}.$$

Furthermore, if $\mathbf{f}$ is a primitive form in $\mathcal{M}_k(\mathfrak{c},\psi)$, we have

$$\mathbf{f}|J_{\mathfrak{c}} = \mathfrak{u}(\mathbf{f})\,\mathbf{f}^{\iota}$$

where $\mathfrak{u}(\mathbf{f})\in\mu_\infty$ is a root of unity, and $\mathbf{f}^\iota$ is the form defined by $C(\mathfrak{m},\mathbf{f}^\iota)=\overline{C(\mathfrak{m},\mathbf{f})}$.

Notations: (i) For $\mathbf{F},\mathbf{G}\in\mathcal{M}_k(\mathfrak{c},\psi)$ one of which is a cusp form, we introduce the (scaled) Petersson inner product via the complex integral

$$\langle\mathbf{F},\mathbf{G}\rangle_{\mathfrak{c}} := \sum_{\lambda=1}^{h}\int_{\Gamma_\lambda(\mathfrak{c})\backslash\mathfrak{H}^d}\overline{\mathbf{F}_\lambda(z)}\mathbf{G}_\lambda(z)N(y)^k d\nu(z)$$

with the choice of hyperbolic metric $d\nu(z)=\prod_{1\le j\le d}y_j^{-2}dx_jdy_j$.

(ii) Let K be a number field. For a finite character $\theta:\mathrm{Gal}(\overline{K}/K)\longrightarrow\mathbb{C}^\times$, write $\theta^\dagger:\mathcal{I}_K\longrightarrow\mathbb{C}^\times$ for the ideal character associated to θ via composition with the reciprocity map – specifically $\theta^\dagger$ is normalised by

$$\theta^\dagger(\mathfrak{q}) \;=\; \theta(\mathrm{Frob}_{\mathfrak{q}})$$

at all primes $\mathfrak{q}$ of K, where $\mathrm{Frob}_{\mathfrak{q}}$ is an arithmetic Frobenius element at $\mathfrak{q}$.

(iii) Given a Hilbert automorphic form $\mathbf{f}$, we shall write $\mathfrak{c}(\mathbf{f})$ to denote its *minimal* level.

The situation relevant to the present paper occurs when one considers a CM extension K/F. In this setting, we have the following key result due to Serre [Ser87].

Theorem 2.2. *Let K/F be a CM extension. If χ is a finite-order Hecke character over K and $\rho=\rho_\chi:=\mathrm{Ind}_K^F(\chi)$, then there exists a Hilbert automorphic form $\mathbf{g}_\rho\in\mathcal{S}_1\big(\mathfrak{c}(\mathbf{g}_\rho),(\det\rho)^\dagger\big)$ over the totally real F, such that*

$$L(\mathbf{g}_\rho,s) \;=\; L(\rho,s) \;=\; L(\chi_{/K},s).$$

Furthermore, $\mathbf{g}_\rho$ is primitive if and only if χ is a primitive character.

Finally if $\mathbf{f}$ and ρ are as in Theorem 2.2, we define the Rankin-Selberg L-function attached to the tensor product $\mathbf{f} \otimes \mathbf{g}_\rho$ by

$$\Psi(\mathbf{f}, \mathbf{g}_\rho, s) \;=\; \left(\frac{\Gamma(s)}{(2\pi)^s}\right)^{2[F:\mathbb{Q}]} \times L_{\mathfrak{c}}\big((\det \rho)^\dagger, 2s-1\big) \cdot L\big(\mathbf{f}, \mathbf{g}_\rho, s\big)$$

where the $\mathcal{O}_F$-ideal $\mathfrak{c}$ is given by $\mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g}_\rho)$, and

$$L(\mathbf{f}, \mathbf{g}_\rho, s) \;=\; \sum_{\mathfrak{a}} C(\mathfrak{a}, \mathbf{f}) C(\mathfrak{a}, \mathbf{g}_\rho) N_{F/\mathbb{Q}}(\mathfrak{a})^{-s}.$$

The analytic continuation and functional equation for the complex L-function $\Psi(\mathbf{f}, \mathbf{g}_\rho, s)$ were established in the work of Klingen and Shimura.

3 The congruence between $R_S(E_\star, \rho_{F_n})$ and $R_S(E_\star, \sigma_{F_n})$

Let $n \geq 1$ be a fixed integer. Recall from the introduction that K_n denotes the number field obtained by adjoining the p^n-th roots of unity μ_{p^n} to $\mathbb{Q}$. For $\star = 1, 2$ we write $\mathbf{f}_\star$ for the base change of $f_\star$ to the totally real field $F_n = K_n^+ = K_n \cap \mathbb{R}$. Since $F_n/\mathbb{Q}$ is an abelian extension, this is the Hilbert modular form whose L-function satisfies

$$L(s, \mathbf{f}_\star) = \prod_{\psi : G_n^+ \to \mathbb{C}^\times} L(s, f_\star, \psi), \qquad \text{where} \;\; G_n^+ = \mathrm{Gal}(F_n/\mathbb{Q}).$$

Let $\chi : \mathrm{Gal}\big(K_n(\Delta^{1/p^n})/K_n\big) \twoheadrightarrow \mu_{p^n}$ be the surjective character determined by $\sigma \mapsto \frac{\sigma(\Delta^{1/p^n})}{\Delta^{1/p^n}}$. One defines $\rho_{F_n} := \mathrm{Ind}_{K_n}^{F_n}(\chi)$. We shall also consider $\sigma_{F_n} = \mathrm{Ind}_{K_n}^{F_n}(\mathbf{1}) = \mathbf{1}_{F_n} \oplus \varepsilon_{K_n/F_n}$, where ε_{K_n/F_n} is the quadratic character attached to the CM extension K_n/F_n.

Our goal is to establish a mod p congruence between the algebraic p-adic L-values attached to the twists $\mathbf{f}_\star \otimes \rho_{F_n} \otimes \psi$ and $\mathbf{f}_\star \otimes \sigma_{F_n} \otimes \psi$, for every finite character $\psi : \mathrm{Gal}(F_\infty/F_n) \to \mathbb{C}^\times$.

3.1 Inner products associated to $\Psi(\mathbf{f}, \mathbf{g}_\rho, 1)$

Throughout this section, we fix a choice of $\star \in \{1, 2\}$ and an integer $n \geq 1$. Recalling that $\mathfrak{p}$ is the unique prime ideal of $\mathcal{O}_{F_n}$ lying above p, let $\alpha_\star(\mathfrak{p})$ and $\beta_\star(\mathfrak{p})$ denote the two roots of

$$X^2 - t_p(E_\star) + p \;=\; X^2 - C(\mathfrak{p}, \mathbf{f}_\star)X + N_{F_n/\mathbb{Q}}(\mathfrak{p}).$$

If $E_\star$ has ordinary reduction at p, we adopt the convention that $\alpha_\star(\mathfrak{p})$ is the p-adic unit root. If $E_\star$ has supersingular reduction at p, then the p-adic valuation of both roots is equal to $1/2$; one may assume that $|\alpha_1(\mathfrak{p}) - \alpha_2(\mathfrak{p})|_p < p^{-1/2}$. For example, if $t_p(E_\star) = 0$ for both $\star = 1, 2$ then we may either take $\alpha_1(\mathfrak{p}) = \alpha_2(\mathfrak{p}) = \sqrt{-p}$, or instead $\alpha_1(\mathfrak{p}) = \alpha_2(\mathfrak{p}) = -\sqrt{-p}$.

Analogously for a finite place $v \neq \mathfrak{p}$ of F_n, we label by $\alpha_\star(v)$, $\beta_\star(v)$ the roots of the polynomial

$$X^2 - C(v, \mathbf{f}_\star)X + N_{F_n/\mathbb{Q}}(v) \;=\; \big(X - \alpha_\star(v)\big)\big(X - \beta_\star(v)\big).$$

One can extend $\alpha_\star(\mathfrak{m})$, $\beta_\star(\mathfrak{m})$ multiplicatively to all ideals $\mathfrak{m}$ of $\mathcal{O}_{F_n}$.

Now consider a finite set of primes S of F_n, containing as a subset

$$S_0 \;=\; \big\{v \;:\; v \text{ is a prime of } F_n,\;\; v\big|\Delta\big\} \cup \big\{\mathfrak{p}\big\}.$$

We denote by $\mathcal{G}_{F_n,S}$ the topological group $\mathrm{Gal}(F_{n,S}^{\mathrm{ab}}/F_n)$, where $F_{n,S}^{\mathrm{ab}}$ is the maximal abelian extension of F_n unramified outside the set S and the infinite places.

Let χ be the character on $\mathrm{Gal}\left(K_n(\Delta^{1/p^n})/K_n\right)$ described above. We can view it both as a character on $\mathcal{G}_{F_n,S}$, and as an ideal character modulo $\mathfrak{f}_\chi \cdot \prod_{\mathfrak{q}\in S-S_0}\mathfrak{q}$. For the rest of this section, $\psi : \mathcal{G}_{F_n,S} \longrightarrow \mathbb{C}^\times$ will denote a fixed finite-order character of conductor $\mathfrak{f}_\psi$.

Definition 3.1. *Set* $\mathfrak{l}_0 := \prod_{\mathfrak{q}\in S-\{\mathfrak{p}\}}\mathfrak{q}$. *For* $j = 1$ *or* 2, *the S-stabilisation of* $\mathbf{f}_\star$ *is defined to be*

$$\mathbf{f}_\star^\alpha \ := \ \sum_{\mathfrak{a}|\mathfrak{p}\mathfrak{l}_0} M(\mathfrak{a})\beta_\star(\mathfrak{a}).\mathbf{f}_\star\big|\mathfrak{a}$$

where M is the Möbius function on ideals.

Suppose either $\rho = \rho_{F_n}\otimes\psi$ or $\rho = \sigma_{F_n,S}\otimes\psi$, where $\sigma_{F_n,S}$ is the representation induced from the trivial character $\mathbf{1}_{K_n}$ modulo $\mathfrak{l}_0\cdot\mathcal{O}_{K_n}$. By Theorem 2.2, there exists $\mathbf{g}_\rho \in \mathcal{M}_1\big(\mathfrak{c}(\mathbf{g}_\rho),\, (\det\rho)^\dagger\big)$ with the same complex L-function as ρ.

Following [Pan91, (3.14)], one may then define the depleted form

$$\mathbf{g}_{\rho,\mathfrak{l}_0} \ := \ \sum_{\mathfrak{n}|\mathfrak{l}_0} M(\mathfrak{n}).\,\mathbf{g}_\rho\big|U(\mathfrak{n})\circ\mathfrak{n}$$

so in particular, $\mathbf{g}_{\rho,\mathfrak{l}_0} \in \mathcal{M}_1\big(\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{l}_0^2,\, (\det\rho)^\dagger\big)$.

We shall choose ideals $\mathfrak{m}'$ and $\mathfrak{l}'$ of $\mathcal{O}_{F_n}$ such that $\mathfrak{m}'$ is a power of $\mathfrak{p}$, $\mathrm{supp}(\mathfrak{l}') = \mathrm{supp}(\mathfrak{l}_0)$, and $\mathfrak{c}(\mathbf{g}_\rho)\cdot\mathfrak{l}_0^2\,\big|\mathfrak{m}'\mathfrak{l}'$ for both $\rho = \rho_{F_n}\otimes\psi$ and $\rho = \sigma_{F_n,S}\otimes\psi$. Setting $\mathfrak{c}_0 := \mathfrak{p}\mathfrak{l}_0\mathfrak{c}(\mathbf{f}_\star)$, it follows that

$$\mathbf{f}_\star^\alpha \ \in \ \mathcal{S}_{\mathbf{2}}(\mathfrak{c}_0) \ \subset \ \mathcal{S}_{\mathbf{2}}\big(\mathfrak{c}(\mathbf{f}_\star)\mathfrak{m}'\mathfrak{l}'\big)$$

and

$$\mathbf{g}_{\rho,\mathfrak{l}_0} \ \in \ \mathcal{M}_{\mathbf{1}}\big(\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{l}_0^2,\,(\det\rho)^\dagger\big) \ \subset \ \mathcal{M}_{\mathbf{1}}\big(\mathfrak{c}(\mathbf{f}_\star)\mathfrak{m}'\mathfrak{l}',(\det\rho)^\dagger\big).$$

Definition 3.2. *We associate a complex linear functional,* $\mathcal{L}^\alpha_{\star,F_n}$, *through the assignment rule*

$$\mathcal{L}^\alpha_{\star,F_n} : \ \Theta \ \longmapsto \ \frac{\big\langle(\mathbf{f}_\star^\alpha)^\iota, \Theta\big|J_{\mathfrak{c}_0}\big\rangle_{\mathfrak{c}_0}}{\langle\mathbf{f}_\star,\mathbf{f}_\star\rangle_{\mathfrak{c}(\mathbf{f}_\star)}}\ .$$

Using the identity $\Phi\Big|U\big(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1}\big)\Big|J_{\mathfrak{c}_0} = \Phi\Big|J_{\mathfrak{c}(\mathbf{f}_\star)\mathfrak{m}'\mathfrak{l}'}\Big|\mathrm{Tr}^{\mathfrak{c}(\mathbf{f}_\star)\mathfrak{m}'\mathfrak{l}'}_{\mathfrak{c}_0}\Big|J_{\mathfrak{c}_0}$ *for every* $\Phi\in\mathcal{M}_{\mathbf{2}}\big(\mathfrak{c}(\mathbf{f}_\star)\mathfrak{m}'\mathfrak{l}'\big)$, *clearly* $\mathcal{L}^\alpha_{\star,F_n}$ *is well-defined on the vector space* $\mathcal{M}_{\mathbf{2}}\big(\mathfrak{c}(\mathbf{f}_\star)\mathfrak{m}'\mathfrak{l}'\big)\Big|U\big(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1}\big)$.

The reason for introducing this functional is that the special value $\Psi(\mathbf{f}_\star,\mathbf{g}_\rho,1)$, which is basically $L(E_\star\,,\rho^*,1)$ multiplied by Euler factors, can be expressed in terms of $\mathcal{L}^\alpha_{\star,F_n}\big(\Phi_\rho^{(n)}\big|U(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1})\big)$ where $\Phi_\rho^{(n)}$ is a product of $\mathbf{g}_{\rho,\mathfrak{l}_0}$ with a Klingen-Eisenstein series.

To be rather more explicit, let θ denote an ideal character modulo $\mathfrak{c}$ for some $\mathcal{O}_{F_n}$-ideal $\mathfrak{c}$. We consider the series $\mathbf{E}_1(0,\,\mathfrak{c},\,\theta)\in\mathcal{M}_1\big(\mathfrak{c},\theta\big)$ whose λ-components are

$$\mathbf{E}_1(0,\,\mathfrak{c},\,\theta)_\lambda(z) \ = \ \frac{N_{F/\mathbb{Q}}(\tilde{t}_\lambda)^{-1/2}D_F^{1/2}}{(-4\pi i)^{[F:\mathbb{Q}]}}\times\sum_{c,d}\ \mathrm{sign}\big(N_{F/\mathbb{Q}}(c)\big)\,\theta(c\,\mathcal{O}_F)\,N_{F/\mathbb{Q}}(cz+d)^{-1}$$

where the summation ranges over pairs $(c,d)\in\frac{\mathcal{O}_F\times\tilde{t}_\lambda^{-1}\mathfrak{d}_F^{-1}}{\sim}$. The Fourier expansion of each λ-component is described in [Pan91, §4, Proposition 4.2], i.e.

$$\mathbf{E}_1(0,\mathfrak{c},\theta)_\lambda(z) \ = \ N_{F/\mathbb{Q}}(\tilde{t}_\lambda)^{-1/2}\times\sum_{0\ll\xi\in\tilde{t}_\lambda}\ \sum_{\substack{\xi=\tilde{b}\tilde{c},\\ c\in\mathcal{O}_F,\\ b\in\tilde{t}_\lambda}}\theta(\tilde{c})\times e_F(\xi z).$$

Definition 3.3. *The parallel weight 2 modular form $\Phi_\rho^{(n)}$ is given by the product*

$$\Phi_\rho^{(n)} \;:=\; \mathbf{g}_{\rho,\mathfrak{l}_0}\cdot \mathbf{E}_1\big(0,\ \mathfrak{c}(\mathbf{f}_\star)\mathfrak{m}'\mathfrak{l}',\ (\det\rho)^{-1}\big)$$

where we again ensure $\mathfrak{m}'$ and $\mathfrak{l}'$ are chosen so that $\mathfrak{c}(\mathbf{g}_\rho)\cdot\mathfrak{l}_0^2$ divides $\mathfrak{m}'\mathfrak{l}'$.

Because we already assumed $E_\star$ is semistable over $\mathbb{Q}$, the Fourier coefficient

$$C\big(\mathfrak{c}(\mathbf{f}_\star),\mathbf{f}_\star\big) \;=\; (-1)^{\#\mathcal{T}^{\mathrm{ns}}_{\star,F_n}}$$

where $\mathcal{T}^{\mathrm{ns}}_{\star,F_n}$ denotes the finite places of F_n at which $E_\star$ has non-split multiplicative reduction; in particular $C\big(\mathfrak{c}(\mathbf{f}_\star),\mathbf{f}_\star\big)\neq 0$, which is crucial to our method.

Notations: (a) If ρ is an Artin representation defined over F_n, we denote by $f(\rho,\mathfrak{p})$ the exponent at $\mathfrak{p}$ occurring in its conductor $\mathfrak{f}_\rho$.

(b) If v is a finite place of F_n not dividing l, we write $P_v(\rho,X)$ for the characteristic polynomial

$$\det\left(1 - X\cdot \mathrm{Frob}_v^{-1}\Big|\big(V_l(\rho)\big)^{I_{F_n,v}}\right)$$

where $V_l(\rho)$ is the l-adic representation space for ρ, the element Frob_v^{-1} denotes a geometric Frobenius for v, and $I_{F_n,v}$ is the inertia subgroup of $\mathrm{Gal}\big(\overline{\mathbb{Q}}/F_n\big)$ at the place v.

(c) Lastly we define a Hecke character $\tilde{\chi}$ over K_n by $\tilde{\chi} = \begin{cases}\chi\otimes\mathrm{Res}_{K_n}(\psi) & \text{if } \rho=\rho_{F_n}\otimes\psi\\ \mathbf{1}_{K_n,\mathfrak{l}_0}\otimes\mathrm{Res}_{K_n}(\psi) & \text{if } \rho=\sigma_{F_n,S}\otimes\psi.\end{cases}$

Theorem 3.4. *If one puts $A_{\tilde{\chi}} = N_{K_n/\mathbb{Q}}(\mathfrak{f}_{\tilde{\chi}})$, then*

$$\frac{i^{h_{F_n}}\,(-4i)^{\phi(p^n)/2}}{\alpha_\star(\mathfrak{m}'\mathfrak{l}')C\big(\mathfrak{c}(\mathbf{f}_\star),\mathbf{f}_\star\big)}\cdot\mathcal{L}^{\alpha}_{\star,F_n}\left(\Phi_\rho^{(n)}\Big|U\big(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1}\big)\right) \;=\; \frac{\epsilon_{F_n}(\rho^*)}{\alpha_\star(p)^{f(\rho,\mathfrak{p})}\prod_{q|\Delta}\alpha_\star(q)^{\mathrm{ord}_q(A_{\tilde{\chi}})}}$$
$$\times\prod_{v|\mathfrak{p}\Delta}\frac{P_v\big(\rho,\alpha_\star(q_v)^{-[\mathcal{F}_{n,v}:\mathbb{F}_{q_v}]}\big)}{P_v\big(\rho^*,\beta_\star(q_v)^{-[\mathcal{F}_{n,v}:\mathbb{F}_{q_v}]}\big)}\times\frac{L_S(E_\star\,,\rho^*,1)}{(2\pi)^{\phi(p^n)}\cdot\langle\mathbf{f}_\star,\mathbf{f}_\star\rangle_{\mathfrak{c}(\mathbf{f}_\star)}}$$

where ρ^* denotes the contragredient representation to ρ.

Proof. The details are almost identical to the demonstration of [DW08, Theorem 3.2]. □

Note that the right-hand side of the above identity does not depend on the $\mathcal{O}_{F_n}$-ideals $\mathfrak{m}'$ or $\mathfrak{l}'$, therefore neither does the left-hand side.

3.2 A specialisation to the cyclotomic line

Henceforth we restrict ourselves to considering only Hecke characters ψ factoring through Γ_n^{cy}, the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of F_n (this simplies most of our formulae).

Lemma 3.5. *If ψ denotes a Hecke character on Γ_n^{cy} of conductor $\mathfrak{p}^m$, then:*

(a) for $m>0$, both polynomials $P_\mathfrak{p}(\rho,X)$ and $P_\mathfrak{p}(\rho^*,X)$ equal one;

(b) for $m\geq 0$ and all places $v\big|\Delta$, both $P_v(\rho,X)$ and $P_v(\rho^*,X)$ equal one;

(c) for all rational primes q dividing Δ, we have $\mathrm{ord}_q(A_{\tilde{\chi}}) = p^n - p^{n-1}$.

Proof. At each finite place v of F_n, the Hecke polynomial of $\mathbf{g}_\rho$ at v admits a factorisation

$$X^2 - C(v, \mathbf{g}_\rho)X + (\det\rho)^\dagger(v) \;=\; \big(X - \lambda(v)\big)\big(X - \mu(v)\big),$$

and likewise the dual Hecke polynomial factorises into

$$X^2 - \overline{C(v, \mathbf{g}_\rho)}X + (\det\rho)^{\dagger\,-1}(v) \;=\; \big(X - \hat{\lambda}(v)\big)\big(X - \hat{\mu}(v)\big).$$

Let us further define $\tau = \rho_{F_n}$ or $\tau = \sigma_{F_n,S}$, depending on whether $\rho = \rho_{F_n} \otimes \psi$ or $\rho = \sigma_{F_n,S} \otimes \psi$. By its very definition

$$P_{\mathfrak{p}}(\tau \otimes \psi^{\pm 1}, X) \;=\; \big(1 - (\psi^{\pm 1})^\dagger(\mathfrak{p})\lambda(\mathfrak{p})X\big) \times \big(1 - (\psi^{\pm 1})^\dagger(\mathfrak{p})\mu(\mathfrak{p})X\big);$$

however ψ is non-trivial if $m > 0$, therefore $(\psi^{\pm 1})^\dagger(\mathfrak{p}) = 0$ which proves (a).

Secondly if ψ is the trivial character and $v\big|\Delta$, then $P_v(\tau, X) = 1$ because each prime v_n of $\mathcal{O}_{K_n}$ above v ramifies in the extension $K_n\big(\Delta^{1/p^n}\big)/K_n$, which means v_n divides the moduli of both Hecke characters χ and $\mathbf{1}_{K_n,\mathfrak{l}_0}$. On the other hand, if ψ is a non-trivial character of $\mathfrak{p}$-power conductor then $\big(V_l(\tau) \otimes \psi^{\pm 1}\big)^{I_{F_n,v}} \cong\; V_l(\tau)^{I_{F_n,v}} \otimes \psi^{\pm 1}$, whence

$$P_v\big(\tau \otimes \psi^{\pm 1}, X\big) \;=\; P_v\big(\tau, \psi(v)^{\pm 1}X\big) \;=\; 1.$$

Finally part (c) was already proven in [DW08, Lemma 3.1]. □

Corollary 3.6. *If either $\rho = \rho_{F_n} \otimes \psi$ or $\rho = \sigma_{F_n,S} \otimes \psi$ with ψ factoring through Γ_n^{cy}, then*

$$R_S\big(E_\star\,, \rho\big) \;=\; \boldsymbol{K}_{\star,F_n} \times \frac{1}{\alpha_\star(\mathfrak{m}'\mathfrak{l}')} \cdot \mathcal{L}^{\alpha}_{\star,F_n}\left(\Phi^{(n)}_\rho \Big| U\big(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1}\big)\right)$$

where the constant $\boldsymbol{K}_{\star,F_n} := \dfrac{i^{h_{F_n}}\,(-4i)^{\phi(p^n)/2}\,\prod_{q|\Delta}\alpha_\star(q)^{p^n-p^{n-1}}}{C\big(\mathfrak{c}(\mathbf{f}_\star),\mathbf{f}_\star\big)} \times \iota_\infty^{-1}\left(\dfrac{\big\langle \mathbf{f}_\star,\mathbf{f}_\star\big\rangle_{\mathfrak{c}(\mathbf{f}_\star)}}{\big(-\Omega^+_{f_\star}\Omega^-_{f_\star}\big)^{[F_n:\mathbb{Q}]}}\right) \in \overline{\mathbb{Q}}.$

Proof. Merging together Theorem 3.4 and Lemma 3.5, and observing that the ratio of Vatsal's period to its automorphic cousin is exactly $\big(-\Omega^+_{f_\star}\Omega^-_{f_\star}\big)^{[F_n:\mathbb{Q}]}$ to $\big\langle \mathbf{f}_\star, \mathbf{f}_\star\big\rangle_{\mathfrak{c}(\mathbf{f}_\star)}$, the result follows. □

Applying Atkin-Lehner theory, we know the linear functional $\mathcal{L}^{\alpha}_{\star,F_n}$ decomposes into a finite linear combination of the Fourier coefficients. Therefore there exist finitely many ideals $\mathfrak{n}_i$ and fixed algebraic numbers $\ell^{\alpha}_{\star,F_n}(\mathfrak{n}_i)$, such that

$$\mathcal{L}^{\alpha}_{\star,F_n}(\Theta) \;=\; \sum_i C(\mathfrak{n}_i, \Theta) \times \ell^{\alpha}_{\star,F_n}(\mathfrak{n}_i) \tag{1}$$

for all elements $\Theta \in \mathcal{M}_2(\mathfrak{c}_0)$.

Let us put $\theta_{K_n,\rho} = \chi$ or $\theta_{K_n,\rho} = \mathbf{1}_{K_n,\mathfrak{l}_0}$, according to whether $\rho = \rho_{F_n} \otimes \psi$ or $\rho = \sigma_{F_n,S} \otimes \psi$. The λ-component of each Fourier coefficient $C\big(\xi, \Phi^{(n)}_\rho\big)$ is then equal to the finite summation

$$\sum_{\xi=\xi_1+\xi_2}\;\sum_{\substack{\mathfrak{a}\lhd\mathcal{O}_K,\\ \mathfrak{a}\bar{\mathfrak{a}}=\xi_1\tilde{t}_\lambda^{-1}}} \theta^\dagger_{K_n,\rho}(\mathfrak{a}) \cdot \psi^\dagger(\xi_1\tilde{t}_\lambda^{-1}) \sum_{\substack{\tilde{\xi_2}=\tilde{b}\tilde{c},\\ c\in\mathcal{O}_{F_n},\\ b\in\tilde{t}_\lambda}} \Big(\det\big(\mathrm{Ind}^{F_n}_{K_n}(\theta_{K_n,\rho})\big)^\dagger\Big)^{-1}(\tilde{c}) \cdot \psi^\dagger(\tilde{c})^{-2}.$$

It follows immediately that for all $\mathfrak{n} \lhd \mathcal{O}_{F_n}$,

$$C\big(\mathfrak{n}, \Phi^{(n)}_{\rho_{F_n}\otimes\psi}\big) \equiv\; C\big(\mathfrak{n}, \Phi^{(n)}_{\sigma_{F_n,S}\otimes\psi}\big) \mod \mathfrak{M}_{\mathbb{C}_p} \tag{2}$$

as the individual values of $\theta_{K_n,\rho} = \chi$ and $\theta_{K_n,\rho} = \mathbf{1}_{K_n,\mathfrak{l}_0}$ differ by at most an element of the maximal ideal of $\mathbb{C}_p$; in other words, one has $\left|\chi(\mathfrak{a}) - \mathbf{1}_{K_n,\mathfrak{l}_0}(\mathfrak{a})\right|_p < 1$ for all ideals $\mathfrak{a}$ of $\mathcal{O}_{K_n}$. Combining Equations (1) and (2) with Corollary 3.6 above, we deduce that

$$\Big|R_S\big(E_\star\,, \rho_{F_n}\otimes\psi\big) - R_S\big(E_\star\,, \sigma_{F_n,S}\otimes\psi\big)\Big|_p \;<\; p^{\mathrm{ord}_p\left(\alpha_\star(\mathfrak{m}')\right)} \times \max\left\{\mathcal{T}^{(\mathfrak{m}'\mathfrak{l}')}_{\star,F_n}\big(\rho_{F_n}\big), \mathcal{T}^{(\mathfrak{m}'\mathfrak{l}')}_{\star,F_n}\big(\sigma_{F_n,S}\big)\right\}$$

where for either $\varrho = \rho_{F_n}$ or $\varrho = \sigma_{F_n,S}$, the respective p-integrality bound is given by

$$\mathcal{T}^{(\mathfrak{m}'\mathfrak{l}')}_{\star,F_n}(\varrho) \;:=\; \Big|\mathbf{K}_{\star,F_n}\Big|_p \times \sup\left\{\left|\mathcal{L}^{\alpha}_{\star,F_n}\left(\Phi^{(n)}_{\varrho\otimes\psi}\Big|U\big(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1}\big)\right)\right|_p \text{ for each } \psi : \Gamma_n^{\mathrm{cy}} \to \mathbb{C}^\times\right\}. \quad (3)$$

Finally, let us now make the minimal choices of exponent

$$\mathfrak{m}' = \mathfrak{p}^{f(\rho_{F_n}\otimes\psi,\mathfrak{p})} \quad \text{and} \quad \mathfrak{l}' = \mathfrak{c}(\mathbf{g}_{\rho_{F_n}\otimes\psi}) \cdot \mathfrak{p}^{-f(\rho_{F_n}\otimes\psi,\mathfrak{p})} \cdot \mathfrak{l}_0^2.$$

Setting $\mathcal{T}_{\star,F_n} := \max\left\{\mathcal{T}^{(\mathfrak{m}'\mathfrak{l}')}_{\star,F_n}\big(\rho_{F_n}\big), \mathcal{T}^{(\mathfrak{m}'\mathfrak{l}')}_{\star,F_n}\big(\sigma_{F_n,S}\big)\right\}$, we have shown the following

Theorem 3.7. *For each character ψ on Γ_n^{cy} and $\star \in \{1,2\}$, if the constant $\mathcal{T}_{\star,F_n} \le 1$ then*

$$\Big|R_S\big(E_\star\,, \rho_{F_n}\otimes\psi\big) - R_S\big(E_\star\,, \sigma_{F_n,S}\otimes\psi\big)\Big|_p \;<\; p^{\mathrm{ord}_p(\alpha_\star(\mathfrak{p}))\times f(\rho_{F_n}\otimes\psi,\mathfrak{p})}.$$

In Section 5.1 we will discuss how, under a certain Hypothesis $(\mathrm{MS})_p$, it follows that $\mathcal{T}_{\star,F_n} \le 1$. In particular, if $E_\star$ has good ordinary reduction at p then $\alpha_\star(\mathfrak{p}) \in \mathbb{Z}_p^\times$; hence

$$\big|R_S\big(E_\star\,, \rho_{F_n}\otimes\psi\big) - R_S\big(E_\star\,, \sigma_{F_n,S}\otimes\psi\big)\big|_p \;<\; 1. \quad (4)$$

If $E_\star$ has good supersingular reduction at p, then $p^{\mathrm{ord}_p(\alpha_\star(\mathfrak{p}))\times f(\rho_{F_n}\otimes\psi,\mathfrak{p})} = p^{f\left(\mathrm{Ind}^{\mathbb{Q}}_{F_n}(\rho_{F_n}\otimes\psi),\, p\right)/2}$ since $\left|\alpha_\star(\mathfrak{p})\right|_p = p^{-1/2}$, and the exponent grows rapidly with the conductor of the character ψ. Moreover under Hypothesis (Vat) stated in §4.1, we show in Theorem 4.5(ii,iii) that for $\psi = \mathbf{1}$,

$$\Big|R_S\big(E_\star\,, \sigma_{F_n,S}\big)\Big|_p \;\le\; p^{f(\sigma_{F_n},\mathfrak{p})/2+1}\,.$$

In fact, $p^{f(\sigma_{F_n},\mathfrak{p})} = \left|\mathrm{Disc}_{\mathbb{Q}(\mu_{p^n})}\right| = p^{np^n-(n+1)p^{n-1}}$ hence in the supersingular case, one deduces:

(3.2.1) if $\big|R_S\big(E_\star\,, \rho_{F_n}\big)\big|_p = \big|R_S\big(E_\star\,, \sigma_{F_n,S}\big)\big|_p$ then

$$\big|R_S\big(E_\star\,, \rho_{F_n}\big) - R_S\big(E_\star\,, \sigma_{F_n,S}\big)\big|_p \;<\; p^{(np^n-(n+1)p^{n-1}+2)/2}\,;$$

(3.2.2) if $\big|R_S\big(E_\star\,, \rho_{F_n}\big)\big|_p < \big|R_S\big(E_\star\,, \sigma_{F_n,S}\big)\big|_p$ then

$$\big|R_S\big(E_\star\,, \rho_{F_n}\big) - R_S\big(E_\star\,, \sigma_{F_n,S}\big)\big|_p \;\le\; p^{(np^n-(n+1)p^{n-1}+2)/2}\,;$$

(3.2.3) if $\big|R_S\big(E_\star\,, \rho_{F_n}\big)\big|_p > \big|R_S\big(E_\star\,, \sigma_{F_n,S}\big)\big|_p$ then

$$\big|R_S\big(E_\star\,, \rho_{F_n}\big) - R_S\big(E_\star\,, \sigma_{F_n,S}\big)\big|_p \;<\; p^{f(\rho_{F_n},\mathfrak{p})/2}\,.$$

Note that the power of p on the right-side of (3.2.3) is given by $p^{f(\rho_{F_n},\mathfrak{p})/2} = \big|\mathrm{Disc}_{\mathbb{Q}(\Delta^{1/p^n})}\big|_p^{-1/2}$.

3.3 Computing L-values in the supersingular case

Let E be an elliptic curve over $\mathbb{Q}$. Recall p was an odd prime, and $\Delta > 1$ a p-power free integer. For any self-dual Artin representation ϱ which factors through $\mathbb{Q}(\mu_p, \sqrt[p]{\Delta})/\mathbb{Q}$, we will consider

$$\mathcal{L}_E(\varrho) \; := \; \frac{\epsilon_p(\varrho)}{\alpha_p^{f(\varrho,p)}} \times \frac{P_p(\varrho, \alpha_p^{-1})}{P_p(\varrho, \beta_p'^{-1})} \cdot \frac{L_{(p\Delta)}(E, \varrho, 1)}{\big(\Omega_+(E)\Omega_-(E)\big)^{\dim(\varrho)/2}} \, .$$

Here $\epsilon_p(\varrho)$ is the local epsilon factor of ϱ at p. We again define α_p, β_p to be the roots of the local polynomial $P_p(E, X) = X^2 - t_p(E)X + p$; when E has good ordinary reduction at p, we always choose α_p to be the root which is a p-adic unit.

Throughout this section, we will focus our attention on the two self-dual Artin representations

$$\sigma_1 \; = \; \mathrm{Ind}_{\mathbb{Q}(\mu_p)}^{\mathbb{Q}}(\mathbf{1}) \quad \text{and} \quad \rho_1 \; = \; \mathrm{Ind}_{\mathbb{Q}(\mu_p)}^{\mathbb{Q}}(\chi)$$

where χ is a non-trivial character of $\mathrm{Gal}\big(\mathbb{Q}(\mu_p, \sqrt[p]{\Delta})/\mathbb{Q}(\mu_p)\big)$. In the paper [DD07], at ordinary primes $p \neq 2$ the Dokchitsers compiled extensive numerical evidence supporting the congruence

$$\mathcal{L}_E(\sigma_1) \; \equiv \; \mathcal{L}_E(\rho_1) \quad \bmod p. \tag{5}$$

These experiments were motivated by recent developments in non-commutative Iwasawa theory, since $\mathcal{L}_E(\rho_1)$ is the special value of a p-adic L-function associated to the motive $h^1(E) \otimes \rho_1$. We will establish that this congruence (5) is true in the final section (and a lot more besides), under an assumption on the p-integrality of certain modular symbols – see Theorem 5.6.

Now suppose that E has good supersingular reduction at p. Then Theorem 3.7 implies a congruence holds modulo $p^{-f(\rho_1,p)/2} \cdot \mathfrak{M}_{\mathbb{C}_p}$, as the Néron periods coincide with Vatsal's. Given the special values may not be p-adically integral, one instead predicts that

$$\Big|\mathcal{L}_E(\sigma_1) - \mathcal{L}_E(\rho_1)\Big|_p \; \leq \; p^{\frac{f(\rho_1,\,p)-1}{2}}, \quad \text{with} \;\; \big|\mathcal{L}_E(\sigma_1)\big|_p \, , \big|\mathcal{L}_E(\rho_1)\big|_p \; \leq \; p^{f(\rho_1,\,p)/2}.$$

These inequalities mirror what occurs in the p-ordinary situation, and can be further refined.

Lemma 3.8. *(i)* $f(\sigma_1, p) \; = \; p - 2$*;*

(ii) If $\Delta > 1$ *is a* $(p-1)^{\mathrm{st}}$*-power free integer, then* $f(\rho_1, p) = \begin{cases} p & \text{if } \Delta^{p-1} \not\equiv 1 \pmod{p^2} \\ p-2 & \text{if } \Delta^{p-1} \equiv 1 \pmod{p^2}. \end{cases}$

Proof. Part (i) follows from the equality $p^{f(\sigma_1,p)} \; = \; \big|\mathrm{Disc}_{\mathbb{Q}(\mu_p)}\big|$, together with the fact that $\mathrm{Disc}_{\mathbb{Q}(\mu_p)} \; = \; (-1)^{(p-1)/2} \times p^{p-2}$ (e.g. see [Was97, Prop 2.7]). To show part (ii), one simply observes that $\mathrm{cond}(\rho_1) = \big|\mathrm{Disc}_{\mathbb{Q}(\Delta^{1/p})}\big|$; then by the main result of [Wes10],

$$\mathrm{Disc}_{\mathbb{Q}(\Delta^{1/p})} \; = \; \begin{cases} (-1)^{(p-1)/2} \times p^p \times \Delta^{p-1} & \text{if } \Delta^{p-1} \not\equiv 1 \; (\bmod p^2) \\ (-1)^{(p-1)/2} \times p^{p-2} \times \Delta^{p-1} & \text{if } \Delta^{p-1} \equiv 1 \; (\bmod p^2). \end{cases}$$

□

Bearing in mind the upper bounds (3.2.1)-(3.2.3) at $n = 1$, we therefore conjecture

$$\Big|\mathcal{L}_E(\sigma_1) - \mathcal{L}_E(\rho_1)\Big|_p \; \leq \; p^{(p-1)/2} \quad \text{and} \quad \big|\mathcal{L}_E(\sigma_1)\big|_p \, , \; \big|\mathcal{L}_E(\rho_1)\big|_p \; \leq \; p^{p/2}. \tag{6}$$

In fact, if $\mathcal{L}_E(\rho_1) = 0$ then we make a stronger prediction than (3.2.2), namely that

$$\big|\mathcal{L}_E(\sigma_1) - \mathcal{L}_E(\rho_1)\big|_p = \big|\mathcal{L}_E(\sigma_1) \; - \; 0\big|_p \; \leq \; p^{(p-3)/2}. \tag{7}$$

We asked Thomas Ward to check whether the inequalities (6) and (7) hold numerically, and the following results are entirely due to his efforts. The calculation is largely the same as in the ordinary case (which was already carried out in [DW08, §6.1]), and he was able to adapt the methods from [DD07]. In the supersingular case he only considered primes where $t_p(E) = 0$, so $\alpha_p = \sqrt{-p}$ and $\beta_p = -\sqrt{-p}$ are the two roots of the polynomial $X^2 + p$ – they are both non p-adic units, and no longer lie in $\mathbb{Q}_p$.

In MAGMA one creates the field $\mathbb{Q}_p(\pi)$ where π is a fixed root of $X^2 + p$; this field is a totally ramified quadratic extension of $\mathbb{Q}_p$, with uniformising element π. One then computes L-values by setting $\alpha_p = \pi$. He deliberately chose cases in which p, N_E, Δ are pairwise coprime, and such that $L(E, \sigma_1, 1) \neq 0$.

Notation: To calculate the algebraic part of the Artin-twisted L-values, let us write

$$L^*(E, \sigma_1) \ := \ \sqrt{\left|\mathrm{Disc}_{\mathbb{Q}(\mu_p)}\right|} \ \cdot \ \frac{L(E, \sigma_1, 1)}{\big(2\Omega_+(E)\Omega_-(E)\big)^{(p-1)/2}}$$

and

$$L^*(E, \rho_1) \ := \ \sqrt{\left|\mathrm{Disc}_{\mathbb{Q}(\Delta^{1/p})}\right|} \ \cdot \ \frac{L(E, \rho_1, 1)}{\big(2\Omega_+(E)\Omega_-(E)\big)^{(p-1)/2}}$$

which are rational numbers; one can then compute $\mathcal{L}_E(\sigma_1)$ and $\mathcal{L}_E(\rho_1)$ as elements of $\mathbb{Q}_p(\pi)$.

When both $\mathcal{L}_E(\sigma_1)$ and $\mathcal{L}_E(\rho_1)$ are p-integral, the numerical evidence supports that they are congruent modulo π. Likewise if they are non-p-integral, we found that (6) and (7) are satisfied, for each elliptic curve E and supersingular prime p considered here.

Unfortunately there are serious computational difficulties involved in calculating $L(E, \rho_1, 1)$. The conductor of this Artin-twist is given by $N(E, \rho_1) = N_E^{p-1} \cdot N_{\rho_1}^2$ (since N_E and N_{ρ_1} were chosen to be coprime) which grows very quickly with the prime p, and this dramatically slows down the computation. For this reason the tables are restricted mostly to the case $p = 3$, although there is one example exhibited for $p = 5$.

Table 1 lists the value of $L^*(E, \sigma_1)$ for each case of E and p (which does not depend on Δ).

Table 1: Values of $L^*(E, \sigma_1)$

p	E	$L^*(E, \sigma_1)$
3	$17A(1)$	2^{-2}
3	$32A(1)$	2^{-2}
3	$56A(1)$	2^{-2}
3	$62A(1)$	2^{-2}
3	$80A(1)$	2^{-1}
3	$161A(1)$	2^{-1}
3	$182A(1)$	5
3	$200C(1)$	2
5	$14A(1)$	2^{-1}

Tables 2, 3, 4, 5 6, 7, 8 and 9 give the results for $p = 3$. Table 10 gives the example for $p = 5$.

Table 2: $p = 3, \quad E = 17A(1) : y^2 + xy + y = x^3 - x^2 - x - 14$

Δ	$L^*(E, \rho_1)$	$\mathcal{L}_E(\sigma_1)$	$\mathcal{L}_E(\rho_1)$
2	2^2	$2.\pi^{-3} + 2.\pi^{-2} + O(\pi^3)$	$2.\pi^{-3} + 2.\pi^{-1} + O(\pi^3)$
5	2^2	$2.\pi^{-3} + 2.\pi^{-2} + O(\pi^1)$	$2.\pi^{-3} + O(\pi^1)$
7	2^2	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$	$1.\pi^{-3} + 2.\pi^{-1} + O(\pi^1)$
10	2^2	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$
11	0	$1.\pi + 1.\pi^2 + 2.\pi^3 + O(\pi^4)$	0
13	2^6	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$1.\pi^{-3} + 2.\pi^{-1} + O(\pi^1)$
14	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + O(\pi^1)$
19	0	$1.\pi + 1.\pi^2 + O(\pi^7)$	0
22	$2^2 3^4$	$2.\pi + 2.\pi^2 + O(\pi^3)$	$1.\pi^5 + O(\pi^7)$
23	2^4	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + 1.\pi^{-1} + O(\pi^1)$
26	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + 2.\pi^{-2} + O(\pi^1)$
29	0	$1.\pi^3 + 1.\pi^4 + O(\pi^7)$	0
31	2^2	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$	$\pi^{-3} + 1.\pi^{-1} + O(\pi^3)$
35	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + 2.\pi^{-2} + O(\pi^3)$
37	2^2	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$
38	$2^2.3^2$	$2.\pi + 2.\pi^2 + O(\pi^7)$	$2.\pi + 2.\pi^3 + O(\pi^5)$
41	$2^4.3^2$	$2.\pi^3 + 2.\pi^4 + O(\pi^5)$	$2.\pi + 1.\pi^3 + O(\pi^5)$

Table 3: $p = 3$, $E = 32A(1)$: $y^2 = x^3 + 4x$

Δ	$L^*(E,\rho_1)$	$\mathcal{L}_E(\sigma_1)$	$\mathcal{L}_E(\rho_1)$
5	2^4	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^1 + O(\pi^4)$	$2.\pi^{-3} + 1.\pi^{-1} + 2.\pi^3 + O(\pi^4)$
7	2^2	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 2.\pi^{-1} + 2.\pi^3 + O(\pi^4)$
11	0	$1.\pi^1 + 1.\pi^2 + 2.\pi^3 + O(\pi^8)$	0
13	2^4	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 1.\pi^1 + O(\pi^4)$
17	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^1 + O(\pi^4)$
19	2^2	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$
23	0	$1.\pi^1 + 1.\pi^2 + 2.\pi^3 + O(\pi^8)$	0
29	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^1 + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-1} + 1.\pi^3 + O(\pi^4)$
31	2^4	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^1 + O(\pi^4)$	$1.\pi^{-3} + 2.\pi^1 + 2.\pi^3 + O(\pi^4)$
35	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^3 + O(\pi^4)$
37	2^2	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$
41	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 1.\pi^1 + 1.\pi^3 + O(\pi^4)$
43	2^4	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 1.\pi^{-1} + 1.\pi^3 + O(\pi^4)$
47	0	$1.\pi^1 + 1.\pi^2 + 2.\pi^5 + O(\pi^8)$	0

Table 4: $p = 3$, $E = 56A(1)$: $y^2 = x^3 + x + 2$

Δ	$L^*(E,\rho_1)$	$\mathcal{L}_E(\sigma_1)$	$\mathcal{L}_E(\rho_1)$
5	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^1 + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^1 + 2.\pi^3 + O(\pi^4)$
11	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-1} + 2.\pi^1 + O(\pi^4)$
13	0	$1.\pi^1 + 1.\pi^2 + 1.\pi^5 + O(\pi^8)$	0
17	0	$2.\pi^1 + 2.\pi^2 + 2.\pi^3 + O(\pi^8)$	0
19	0	$1.\pi^1 + 1.\pi^2 + 1.\pi^3 + O(\pi^8)$	0
23	0	$1.\pi^1 + 1.\pi^2 + 2.\pi^3 + O(\pi^8)$	0
29	$2^2.3^2$	$1.\pi^3 + 1.\pi^4 + 2.\pi^7 + O(\pi^{10})$	$2.\pi^1 + 2.\pi^3 + 1.\pi^7 + O(\pi^8)$
31	$2^2.3^2$	$1.\pi^1 + 1.\pi^2 + 2.\pi^3 + O(\pi^8)$	$1.\pi^1 + 1.\pi^3 + 1.\pi^7 + O(\pi^8)$
37	2^2	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$
41	2^4	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^1 + O(\pi^4)$	$2.\pi^{-3} + 1.\pi^{-1} + 2.\pi^1 + O(\pi^4)$
43	0	$1.\pi^1 + 1.\pi^2 + 2.\pi^7 + O(\pi^8)$	0
47	2^4	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^1 + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^1 + O(\pi^4)$

Table 5: $p = 3, \quad E = 62A(1) : y^2 + xy + y = x^3 - x^2 - x + 1$

Δ	$L^*(E,\rho_1)$	$\mathcal{L}_E(\sigma_1)$	$\mathcal{L}_E(\rho_1)$
5	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^1 + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^1 + 2.\pi^3 + O(\pi^4)$
7	2^4	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 1.\pi^{-1} + 1.\pi^1 + O(\pi^4)$
11	$2^2.3^2$	$1.\pi^1 + 1.\pi^2 + 2.\pi^3 + O(\pi^8)$	$2.\pi^1 + 2.\pi^3 + 2.\pi^5 + O(\pi^8)$
13	0	$1.\pi^1 + 1.\pi^2 + 1.\pi^5 + O(\pi^8)$	0
17	0	$2.\pi^1 + 2.\pi^2 + 2.\pi^3 + O(\pi^8)$	0
19	2^2	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$
23	2^4	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 1.\pi^{-1} + 1.\pi^1 + O(\pi^4)$
29	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-1} + 1.\pi^3 + O(\pi^4)$
35	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^3 + O(\pi^4)$
37	2^4	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^1 + O(\pi^4)$	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^3 + O(\pi^4)$
41	0	$2.\pi^3 + 2.\pi^4 + 2.\pi^5 + O(\pi^{10})$	0
43	$2^2.3^2$	$1.\pi^5 + 1.\pi^6 + 2.\pi^7 + O(\pi^{12})$	$1.\pi^1 + 2.\pi^3 + 2.\pi^5 + O(\pi^8)$
47	2^2	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^1 + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-1} + 1.\pi^1 + O(\pi^4)$

Table 6: $p = 3, \quad E = 80A(1) : y^2 = x^3 - 7x + 6$

Δ	$L^*(E,\rho_1)$	$\mathcal{L}_E(\sigma_1)$	$\mathcal{L}_E(\rho_1)$
7	2^3	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 1.\pi^{-1} + 2.\pi^1 + O(\pi^4)$
11	2^3	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^3 + O(\pi^4)$	$1.\pi^{-3} + 2.\pi^3 + O(\pi^4)$
13	2^3	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-1} + 1.\pi^1 + O(\pi^4)$
17	2^3	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^3 + O(\pi^4)$	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$
19	0	$2.\pi^1 + 2.\pi^2 + 1.\pi^7 + O(\pi^8)$	0
23	2^3	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 2.\pi^{-1} + 1.\pi^1 + O(\pi^4)$
29	2^9	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + O(\pi^4)$
31	0	$2.\pi^1 + 2.\pi^2 + 1.\pi^3 + O(\pi^8)$	0
37	2^3	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$
41	$2^3.3^4$	$1.\pi^3 + 1.\pi^4 + 1.\pi^9 + O(\pi^{10})$	$1.\pi^5 + 2.\pi^7 + 1.\pi^{11} + O(\pi^{12})$
43	$2^3.3^2$	$2.\pi^5 + 2.\pi^6 + 1.\pi^7 + O(\pi^{12})$	$2.\pi^1 + 1.\pi^3 + 2.\pi^7 + O(\pi^8)$

Table 7: $p = 3$, $E = 161A(1) : y^2 + xy + y = x^3 - x^2 - 9x + 8$

Δ	$L^*(E, \rho_1)$	$\mathcal{L}_E(\sigma_1)$	$\mathcal{L}_E(\rho_1)$
2	2	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$	$1.\pi^{-3} + 1.\pi^{-1} + O(\pi^2)$
5	2^3	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$	$1.\pi^{-3} + 2.\pi^{-1} + O(\pi^1)$
10	2	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$
11	2^3	$1.\pi^{-3} + 1.\pi^{-2} + O(\pi^0)$	$1.\pi^{-3} + 1.\pi^{-2} + O(\pi^0)$
13	2^3	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + 2.\pi^{-1} + O(\pi^1)$
17	2	$1.\pi^{-3} + 1.\pi^{-2} + O(\pi^3)$	$1.\pi^{-3} + 1.\pi^{-2} + O(\pi^1)$
19	2	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$
22	2^5	$2.\pi^{-3} + 2.\pi^{-2} + O(\pi^3)$	$2.\pi^{-3} + O(\pi^3)$
26	2	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$1.\pi^{-3} + 1.\pi^{-2} + O(\pi^1)$
29	2^5	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$1.\pi^{-3} + 2.\pi^{-1} + O(\pi^4)$
31	2.3^2	$2.\pi^5 + 2.\pi^6 + O(\pi^{11})$	$2.\pi + 2.\pi^3 + O(\pi^7)$
34	2^5	$2.\pi^{-3} + 2.\pi^{-2} + O(\pi^3)$	$2.\pi^{-3} + 2.\pi^{-1} + O(\pi^1)$
37	2	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$
38	2^7	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$1.\pi^{-3} + 1.\pi^{-1} + O(\pi^1)$

Table 8: $p = 3$, $E = 182A(1) : y^2 + xy + y = x^3 - x^2 + 866x + 6445$

Δ	$L^*(E, \rho_1)$	$\mathcal{L}_E(\sigma_1)$	$\mathcal{L}_E(\rho_1)$
5	$2^4.5$	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$	$1.\pi^{-3} + 2.\pi^{-1} + O(\pi^3)$
11	$2^2.5.7^2$	$1.\pi^{-3} + 1.\pi^{-2} + O(\pi^1)$	$1.\pi^{-3} + O(\pi^3)$
17	$2^2.3^25$	$1.\pi^{-3} + 1.\pi^{-2} + O(\pi^3)$	$1.\pi^{-3} + 1.\pi^{-2} + O(\pi^1)$
19	$2^4.5$	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$
23	$2^2.5^3$	$1.\pi^{-3} + 1.\pi^{-2} + O(\pi^1)$	$1.\pi^{-3} + 1.\pi^{-1} + O(\pi^1)$
29	$2^2.5$	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$	$1.\pi^{-3} + 1.\pi^{-1} + O(\pi^1)$
31	$2^8.5$	$2.\pi^{-3} + 2.\pi^{-2} + O(\pi^3)$	$2.\pi^{-3} + 1.\pi^{-1} + O(\pi^1)$
37	$2^4.5$	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^0)$
41	$2^6.3^2.5$	$1.\pi^3 + 1.\pi^4 + O(\pi^6)$	$1.\pi + 1.\pi^3 + O(\pi^5)$
43	$2^6.5$	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^0)$	$2.\pi^{-3} + 2.\pi^{-1} + O(\pi^1)$

Table 9: $p = 3$, $E = 200C(1) : y^2 = x^3 - 50x + 125$

Δ	$L^*(E, \rho_1)$	$\mathcal{L}_E(\sigma_1)$	$\mathcal{L}_E(\rho_1)$
7	2^3	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 1.\pi^{-1} + 2.\pi^1 + O(\pi^4)$
11	2^3	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 2.\pi^3 + O(\pi^4)$
13	0	$2.\pi^1 + 2.\pi^2 + 1.\pi^3 + O(\pi^8)$	0
17	2^3	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$1.\pi^{-3} + 1.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$
19	2^3	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-2} + 2.\pi^{-1} + O(\pi^4)$
23	2^3	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^1 + O(\pi^4)$	$1.\pi^{-3} + 2.\pi^{-1} + 1.\pi^1 + O(\pi^4)$
29	2^5	$1.\pi^{-3} + 1.\pi^{-2} + 1.\pi^1 + O(\pi^4)$	$1.\pi^{-3} + 2\ .\pi^{-1} + O(\pi^4)$
31	2^7	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 1.\pi^{-1} + 2.\pi^1 + O(\pi^4)$
37	2^5	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^{-1} + O(\pi^4)$	$2.\pi^{-3} + 2.\pi^{-2} + 1.\pi^3 + O(\pi^4)$

Table 10: $p = 5$, $E = 14A(1) : y^2 + xy + y = x^3 + 4x - 6$

Δ	$L^*(E, \rho_1)$	$\mathcal{L}_E(\sigma_1)$	$\mathcal{L}_E(\rho_1)$
3	$2^3.3^2$	$2.\pi^{-5} + 1.\pi^{-4} + 1.\pi^{-3} + O(\pi^{-2})$	$2.\pi^{-5} + 4.\pi^{-3} + O(\pi^{-1})$

4 The congruence between $R_S(E_1, \sigma_{F_n})$ and $R_S(E_2, \sigma_{F_n})$

The main obstacle that prevents us relating $R_S(E_1, \rho_{F_n})$ and $R_S(E_2, \rho_{F_n})$ p-adically is that the representation $\rho_n = \mathrm{Ind}_{F_n}^{\mathbb{Q}}(\rho_{F_n})$ is irreducible, so the ρ_n-twisted L-functions do not split. However, because $R_S(E_\star, \rho_{F_n})$ and $R_S(E_\star, \sigma_{F_n})$ are already connected via a mod p congruence and as the induced representation $\sigma_n = \mathrm{Ind}_{F_n}^{\mathbb{Q}}(\sigma_{F_n})$ splits completely into a sum of characters, the corresponding p-adic L-values $R_S(E_\star, \sigma_{F_n})$ decompose into a $\phi(p^n)$-fold product "$\prod_\eta \ldots$".

One can therefore obtain the desired mod p congruence indirectly, by instead showing that each η-component of $R_S(E_1, \sigma_{F_n})$ and $R_S(E_2, \sigma_{F_n})$ in the product decomposition is congruent. We begin by recalling the method of Vatsal [Vat99], which works for classical cusp forms over $\mathbb{Q}$. We next use his theorem repeatedly, to obtain a mod p congruence for each of the η-twists.

4.1 A review of Vatsal's results

Fix an integer $M = Np^s \geq 4$, with N coprime to p. Let $\mathcal{F} = \sum t_m(\mathcal{F})q^m$ and $\mathcal{G} = \sum t_m(\mathcal{G})q^m$ be normalized Hecke eigenforms of weight $k \geq 2$ on the congruence subgroup $\Gamma_1(M) \subset \mathrm{SL}_2(\mathbb{Z})$. We choose a finite extension $K/\mathbb{Q}_p$, whose ring of integers $\mathcal{O}_K$ contains $\{t_m(\mathcal{F})\}_m \cup \{t_m(\mathcal{G})\}_m$. If π is a uniformizer of $\mathcal{O}_K$, we suppose that

$$t_m(\mathcal{F}) \equiv t_m(\mathcal{G}) \pmod{\pi^r} \quad \text{for every } m \geq 1,$$

and for a fixed $r \geq 1$; thus it makes sense to write $\mathcal{F} \equiv \mathcal{G} \pmod{\pi^r}$.

Let $\mathbf{h}_k = \mathbf{h}_k(\Gamma_1(M), \mathcal{O}_K)$ denote the $\mathcal{O}_K$-algebra generated by the Hecke operators T_n, acting on the cusp forms $\mathcal{S}_k(\Gamma_1(M), \mathcal{O}_K)$. The congruence class of $\mathcal{F}$ and $\mathcal{G}$ determines a maximal ideal $\mathfrak{m}$ of $\mathbf{h}_k(\Gamma_1(M), \mathcal{O}_K)$, and a residual representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2\left(\mathbf{h}_k(\Gamma_1(M), \mathcal{O}_K)/\mathfrak{m}\right)$$

such that $\mathrm{Tr}(\rho_{\mathfrak{m}}(\mathrm{Frob}_l)) = T_l$ at all primes $l \nmid Np$.

Remark: Assume now that $\rho_{\mathfrak{m}}$ is irreducible. The localised ring $\mathbf{h}_{k,\mathfrak{m}} \otimes \mathbb{Q}$ is Artinian, and there exist surjective K-algebra mappings $\pi_{\mathcal{F}}, \pi_{\mathcal{G}} : \mathbf{h}_{k,\mathfrak{m}} \otimes \mathbb{Q} \twoheadrightarrow K$ corresponding to both $\mathcal{F}$ and $\mathcal{G}$. As $\mathbf{h}_{k,\mathfrak{m}} \otimes \mathbb{Q}$ decomposes into a direct product of local factors, we shall denote by $R_{\mathcal{F}}$ and $R_{\mathcal{G}}$ the respective components through which $\pi_{\mathcal{F}}$ and $\pi_{\mathcal{G}}$ factor.

We write $\mathrm{Sym}_n(\mathcal{O}_K)$ for the homogeneous polynomials of degree n, with coefficients in $\mathcal{O}_K$. The Hecke algebra $\mathbf{h}_k(\Gamma_1(M), \mathcal{O}_K)$ acts naturally on $H^1(\Gamma_1(M), \mathrm{Sym}_{k-2}(\mathcal{O}_K))$, and also on the parabolic cohomology $H^1_P(\Gamma_1(M), \mathrm{Sym}_{k-2}(\mathcal{O}_K))$; we can decompose both these cohomology groups into $\pm$-eigenspaces under the action of complex conjugation.

Henceforth we shall assume the following conditions from [Vat99] hold for $\mathcal{F}$ and $\mathcal{G}$.

Hypothesis (Vat). *(i) The residual representation $\rho_{\mathfrak{m}}$ is irreducible;*

(ii) the local factors satisfy $R_{\mathcal{F}} = R_{\mathcal{G}} = K$;

(iii) there are isomorphisms of $\mathbf{h}_k(\Gamma_1(M), \mathcal{O}_K)$-modules,

$$H^1(\Gamma_1(M), \mathrm{Sym}_{k-2}(\mathcal{O}_K))^{\pm}_{\mathfrak{m}} = H^1_P(\Gamma_1(M), \mathrm{Sym}_{k-2}(\mathcal{O}_K))^{\pm}_{\mathfrak{m}} \cong \mathbf{h}^*_{2,\mathfrak{m}} = \mathrm{Hom}_{\mathcal{O}_K}(\mathbf{h}_{2,\mathfrak{m}}, \mathcal{O}_K)_{\mathfrak{m}};$$

(iv) in particular, $H^1(\Gamma_1(M), \mathrm{Sym}_{k-2}(\mathcal{O}_K))$ is torsion-free.

As explained in §1.3 of *op. cit.* there exist canonical periods $\Omega^{\pm}_{\mathcal{F}}$ and $\Omega^{\pm}_{\mathcal{G}}$, obtained from cutting out the $\mathcal{F}$- and $\mathcal{G}$- isotypic pieces from the parabolic cohomology of the modular curve $X_1(M)$. At weight $k = 2$ and trivial nebentypus, they agree (up to a unit) with the usual Néron periods $\Omega_{\pm}(E_{\mathcal{F}})$ and $\Omega_{\pm}(E_{\mathcal{G}})$, associated to the modular elliptic curves $E_{\mathcal{F}}, E_{\mathcal{G}} \subset \mathrm{Jac}(X_1(M))$.
The following result is Proposition 1.7 and §1.9 in *op. cit.*

Proposition 4.1. *If η is a Dirichlet character, then for all integers m such that $0 \le m \le k-2$:*

$$\tau(\eta^{-1}) \cdot \binom{k-2}{m} \cdot m! \cdot \frac{L(\mathfrak{h}, \eta, m+1)}{(-2\pi i)^{m+1} \Omega_{\mathfrak{h}}^{\mathrm{sign}(\eta)}} \;\in\; \mathcal{O}_K$$

where $\mathfrak{h} = \mathcal{F}$ or $\mathcal{G}$, and $\tau(\eta^{-1})$ denotes the Gauss sum associated to η^{-1}. Furthermore,

$$\tau(\eta^{-1}) \cdot \binom{k-2}{m} \cdot m! \cdot \frac{L(\mathcal{F}, \eta, m+1)}{(-2\pi i)^{m+1} \Omega_{\mathcal{F}}^{\mathrm{sign}(\eta)}} \;\equiv\; \tau(\eta^{-1}) \cdot \binom{k-2}{m} \cdot m! \cdot \frac{L(\mathcal{G}, \eta, m+1)}{(-2\pi i)^{m+1} \Omega_{\mathcal{G}}^{\mathrm{sign}(\eta)}} \mod \pi^r.$$

Corollary 4.2. *If $\mathcal{F}$ and $\mathcal{G}$ are ordinary at p, then*

$$\left(1 - \alpha_{\mathcal{F}}(p)^{-1}\right)^2 \cdot \frac{L(\mathcal{F}, 1)}{-2\pi i \Omega_{\mathcal{F}}^+} \;\equiv\; \left(1 - \alpha_{\mathcal{G}}(p)^{-1}\right)^2 \cdot \frac{L(\mathcal{G}, 1)}{-2\pi i \Omega_{\mathcal{G}}^+} \mod \pi^r$$

where $\alpha_{\mathcal{F}}(p)$ and $\alpha_{\mathcal{G}}(p)$ denote the p-adic unit roots for the respective Hecke polynomials at p.

Proof. Let $\mathbf{L}_p(\mathcal{F})$ and $\mathbf{L}_p(\mathcal{G})$ be the p-adic L-functions over the $\mathbb{Z}_p$-cyclotomic extension of $\mathbb{Q}$, associated to $\mathcal{F}$ and $\mathcal{G}$ respectively. Using [Vat99, Theorem 1.10], then Proposition 4.1 implies

$$\mathbf{L}_p(\mathcal{F}) \;\equiv\; \mathbf{L}_p(\mathcal{G}) \mod \pi^r \mathcal{O}_K[\![X]\!].$$

The result follows upon evaluating both $\mathbf{L}_p(\mathcal{F})$ and $\mathbf{L}_p(\mathcal{G})$ at the trivial character $\eta = \mathbf{1}$. □

The situation is quite different in the non-ordinary case, because the p-adic L-functions are not defined over $\mathcal{O}_K[\![X]\!]$, and the quantity $\left(1 - \alpha_{\mathfrak{h}}(p)^{-1}\right)^2 \cdot \frac{L(\mathfrak{h},1)}{-2\pi i \Omega_h^+}$ may now not lie inside $\mathcal{O}_K$. Nevertheless, if w is the p-adic valuation of $\alpha_{\mathfrak{h}}(p)$ where $\mathfrak{h}$ is either $\mathcal{F}$ or $\mathcal{G}$, then

$$\left| \left(1 - \alpha_{\mathfrak{h}}(p)^{-1}\right)^2 \cdot \frac{L(\mathfrak{h}, 1)}{-2\pi i \Omega_{\mathfrak{h}}^+} \right|_p \;\le\; p^{2w}$$

since Proposition 4.1 tells us that $\frac{L(\mathfrak{h},1)}{-2\pi i \Omega_{\mathfrak{h}}^+} \in \mathcal{O}_K$.

Lemma 4.3. *If there exist roots $\alpha_{\mathcal{F}}(p)$ and $\alpha_{\mathcal{G}}(p)$ of the Hecke polynomials for $\mathcal{F}$ and $\mathcal{G}$ at p, such that*

$$\left|\alpha_{\mathcal{F}}(p)\right|_p \;=\; \left|\alpha_{\mathcal{G}}(p)\right|_p \;=\; p^{-w} \quad \text{and} \quad \left|\alpha_{\mathcal{F}}(p)^{-1} - \alpha_{\mathcal{G}}(p)^{-1}\right|_p \;\le\; p^{-c}$$

for some real numbers $w, c > 0$, then

$$\left| \left(1 - \alpha_{\mathcal{F}}(p)^{-1}\right)^2 \cdot \frac{L(\mathcal{F}, 1)}{-2\pi i \Omega_{\mathcal{F}}^+} \;-\; \left(1 - \alpha_{\mathcal{G}}(p)^{-1}\right)^2 \cdot \frac{L(\mathcal{G}, 1)}{-2\pi i \Omega_{\mathcal{G}}^+} \right|_p \;\le\; \max\left\{p^{2w - r\cdot \mathrm{ord}_p(\pi)},\; p^{w-c}\right\}.$$

Proof. Let us write $\alpha_{\mathcal{F}}(p)^{-1} = \alpha_{\mathcal{G}}(p)^{-1} + p^c x$ where $x \in \mathcal{O}_{\mathbb{C}_p}$. Expanding the squared factor:

$$\left(1 - \alpha_{\mathcal{F}}(p)^{-1}\right)^2 \cdot \frac{L(\mathcal{F}, 1)}{-2\pi i \Omega_{\mathcal{F}}^+} \;=\; \left(p^{2c}x^2 - 2p^c x\left(1 - \alpha_{\mathcal{G}}(p)^{-1}\right) + \left(1 - \alpha_{\mathcal{G}}(p)^{-1}\right)^2\right) \cdot \frac{L(\mathcal{F}, 1)}{-2\pi i \Omega_{\mathcal{F}}^+}.$$

As a direct consequence,

$$\left| \left(1 - \alpha_{\mathcal{F}}(p)^{-1}\right)^2 \cdot \frac{L(\mathcal{F}, 1)}{-2\pi i \Omega_{\mathcal{F}}^+} - \left(1 - \alpha_{\mathcal{G}}(p)^{-1}\right)^2 \cdot \frac{L(\mathcal{G}, 1)}{-2\pi i \Omega_{\mathcal{G}}^+} \right|_p \;\le\; \max\left\{ \left| p^{2c}x^2 \cdot \frac{L(\mathcal{F}, 1)}{-2\pi i \Omega_{\mathcal{F}}^+} \right|_p, \right.$$

$$\left. \left| -2p^c x\left(1 - \alpha_{\mathcal{G}}(p)^{-1}\right) \cdot \frac{L(\mathcal{F}, 1)}{-2\pi i \Omega_{\mathcal{F}}^+} \right|_p, \; \left| \left(1 - \alpha_{\mathcal{G}}(p)^{-1}\right)^2 \cdot \left(\frac{L(\mathcal{F}, 1)}{-2\pi i \Omega_{\mathcal{F}}^+} - \frac{L(\mathcal{G}, 1)}{-2\pi i \Omega_{\mathcal{G}}^+} \right) \right|_p \right\}$$

which is bounded above by $\max\left\{p^{-2c},\; p^{w-c},\; p^{2w - \mathrm{ord}_p(\pi^r)}\right\}$ upon applying Proposition 4.1. □

4.2 Behaviour under twisting by σ_{F_n}

Let E_1 and E_2 be the two elliptic curves as before. In this section, we shall exclusively consider

$$\rho \;=\; \sigma_{F_n} \otimes \psi \;=\; \left(\mathrm{Ind}_{K_n}^{F_n}(\mathbf{1})\right) \otimes \psi \;=\; \psi \oplus \psi \cdot \varepsilon_{K_n/F_n}$$

where ψ is a character on Γ_n^{cy}. Remember our goal is to study $R_S(E_\star, \rho)$, which is defined to be

$$\frac{\epsilon_{F_n}(\rho^*)}{\alpha_\star(p)^{f(\rho,\mathfrak{p})}} \times \frac{P_{\mathfrak{p}}\left(\rho, \alpha_\star(\mathfrak{p})^{-[\mathcal{F}_{n,\mathfrak{p}}:\mathbb{F}_p]}\right)}{P_{\mathfrak{p}}\left(\rho^*, \beta_\star(\mathfrak{p})^{-[\mathcal{F}_{n,\mathfrak{p}}:\mathbb{F}_p]}\right)} \times \frac{L_S(E_\star, \rho^*, 1)}{\left((2\pi i)^2 \Omega_{f_\star}^+ \Omega_{f_\star}^-\right)^{[F_n:\mathbb{Q}]}}\;.$$

If ψ is a non-trivial character, then $P_{\mathfrak{p}}(\rho, X) = 1$. On the other hand, if ψ is the trivial character then both $P_{\mathfrak{p}}(\rho, X)$ and $P_{\mathfrak{p}}(\rho^*, X)$ are given by the polynomial $(1-X)\left(1-\varepsilon_{K_n/F_n}(\mathfrak{p})X\right) = (1-X)$. Since p is totally ramified in F_n we have $\mathcal{F}_{n,\mathfrak{p}} = \mathbb{F}_p$, in which case

$$\frac{P_{\mathfrak{p}}\left(\rho, \alpha_\star(p)^{-[\mathcal{F}_{n,\mathfrak{p}}:\mathbb{F}_p]}\right)}{P_{\mathfrak{p}}\left(\rho^*, \beta_\star(p)^{-[\mathcal{F}_{n,\mathfrak{p}}:\mathbb{F}_p]}\right)} \;=\; \frac{1-\alpha_\star(p)^{-1}}{1-\beta_\star(p)^{-1}}\;.$$

Corollary 4.4. *If $S' = S \setminus \{\mathfrak{p}\}$, then*

$$R_S(E_\star\,, \rho) \;=\; \begin{cases} \left(1-\alpha_\star(p)^{-1}\right)^2 \times \frac{\epsilon_{F_n}(\rho^*)}{\alpha_\star(p)^{f(\rho,\mathfrak{p})}} \times \frac{L_{S'}(E_\star,\rho^*,1)}{\left((2\pi i)^2 \Omega_{f_\star}^+ \Omega_{f_\star}^-\right)^{[F_n:\mathbb{Q}]}} & \text{if } \psi = \mathbf{1} \\ \frac{\epsilon_{F_n}(\rho^*)}{\alpha_\star(p)^{f(\rho,\mathfrak{p})}} \times \frac{L_{S'}(E_\star,\rho^*,1)}{\left((2\pi i)^2 \Omega_{f_\star}^+ \Omega_{f_\star}^-\right)^{[F_n:\mathbb{Q}]}} & \text{if } \psi \neq \mathbf{1}. \end{cases}$$

Recall that f_1 and f_2 are the weight two forms corresponding to the elliptic curves E_1 and E_2. We now assume the Fourier coefficients satisfy $t_m(E_1) \equiv t_m(E_2) \pmod{p^r}$ if $\gcd(m, N_1N_2) = 1$. If one puts $M = \prod_{l \mid N_1N_2} l^{1+\max\{\mathrm{ord}_l(N_1), \mathrm{ord}_l(N_2)\}}$, then the depleted modular forms

$$\mathcal{F} \;=\; \sum_{\gcd(m,N_1N_2)=1} t_m(E_1) \cdot q^m \quad \text{and} \quad \mathcal{G} \;=\; \sum_{\gcd(m,N_1N_2)=1} t_m(E_2) \cdot q^m$$

are both Hecke eigenforms of weight two on $\Gamma_0(M)$, with their Euler factors at $N_1 \cdot N_2$ removed. By their construction $\mathcal{F} \equiv \mathcal{G} \pmod{p^r}$, so one can certainly apply Vatsal's congruences to them. We shall also require the sets S_1 and S_2, defined by

$$S_\star \;=\; \left\{v \;:\; v \text{ is a place of } F_n \text{ satisfying } v \big| N_\star\right\}.$$

Remarks: (a) Let $\sigma_n = \mathrm{Ind}_{F_n}^{\mathbb{Q}}(\rho^*)$; we have a natural decomposition

$$\sigma_n \;=\; \bigoplus_{\eta : G_n \to \mathbb{C}^\times} \eta \otimes \psi_0$$

where $G_n = \mathrm{Gal}(K_n/\mathbb{Q})$, and ψ_0 denotes any character on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\psi_0\big|_{\mathcal{G}_{F_n,S}} = \psi^{-1}$.

(b) It follows immediately that

$$L_{S_1\cup S_2}(E_1, \rho^*, s) \;=\; \prod_\eta L(\mathcal{F}, \psi_0 \otimes \eta, s) \quad \text{and} \quad L_{S_1\cup S_2}(E_2, \rho^*, s) \;=\; \prod_\eta L(\mathcal{G}, \psi_0 \otimes \eta, s)$$

where the product ranges over characters $\eta : G_n \to \mathbb{C}^\times$.

(c) By [SS15, Lemma 4.3], the periods $\Omega^{\pm}_{f_1}$ and $\Omega^{\pm}_{\mathcal{F}}$ differ by a p-adic unit, and likewise $\Omega^{\pm}_{f_2}$ and $\Omega^{\pm}_{\mathcal{G}}$ differ by the same unit; without loss of generality, they may be interchanged as we please.

(d) Lastly by the Artin formalism, the Gauss sum $\epsilon_{F_n}(\rho)$ equals $\prod_{\eta:G_n\to\mathbb{C}^\times}\tau\big((\eta\otimes\psi_0)^{-1}\big)$.

Combining together (a)-(d) with Corollary 4.4, we deduce that

$$R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_1,\rho) \;=\; \frac{1}{\alpha_1(p)^{f(\rho,\mathfrak{p})}}\times R(\mathcal{F},\psi_0)\times\prod_{\eta\neq\mathbf{1}}\frac{\tau\big((\eta\otimes\psi_0)^{-1}\big)\cdot L(\mathcal{F},\eta\otimes\psi_0,1)}{\Big(-2\pi i\Omega_{\mathcal{F}}^{\mathrm{sign}(\eta\psi_0)}\Big)}$$

and

$$R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_2,\rho) \;=\; \frac{1}{\alpha_2(p)^{f(\rho,\mathfrak{p})}}\times R(\mathcal{G},\psi_0)\times\prod_{\eta\neq\mathbf{1}}\frac{\tau\big((\eta\otimes\psi_0)^{-1}\big)\cdot L(\mathcal{G},\eta\otimes\psi_0,1)}{\Big(-2\pi i\Omega_{\mathcal{G}}^{\mathrm{sign}(\eta\psi_0)}\Big)}\;,$$

where $R(\mathfrak{h},\psi_0):=\begin{cases}\big(1-\alpha_{\mathfrak{h}}(p)^{-1}\big)^2\cdot\frac{L(\mathfrak{h},1)}{(-2\pi i\Omega_{\mathfrak{h}}^{+})} & \text{if } \psi=\mathbf{1}\\ \frac{\tau(\psi_0^{-1})\cdot L(\mathfrak{h},\psi_0,1)}{\big(-2\pi i\Omega_{\mathfrak{h}}^{\mathrm{sign}(\psi_0)}\big)} & \text{if } \psi\neq\mathbf{1}.\end{cases}$

Applying Proposition 4.1, for each $\eta: G_n\to\mathbb{C}^\times$ the quantities

$$\tau\big((\eta\otimes\psi_0)^{-1}\big)\cdot\frac{L(\mathcal{F},\eta\otimes\psi_0,1)}{-2\pi i\Omega_{\mathcal{F}}^{\mathrm{sign}(\eta\psi_0)}}\qquad\text{and}\qquad\tau\big((\eta\otimes\psi_0)^{-1}\big)\cdot\frac{L(\mathcal{G},\eta\otimes\psi_0,1)}{-2\pi i\Omega_{\mathcal{G}}^{\mathrm{sign}(\eta\psi_0)}}$$

are p-integral; moreover for ψ non-trivial or $\eta\neq\mathbf{1}$,

$$\tau\big((\eta\otimes\psi_0)^{-1}\big)\cdot\frac{L(\mathcal{F},\eta\otimes\psi_0,1)}{-2\pi i\Omega_{\mathcal{F}}^{\mathrm{sign}(\eta\psi_0)}}\;\equiv\;\tau\big((\eta\otimes\psi_0)^{-1}\big)\cdot\frac{L(\mathcal{G},\eta\otimes\psi_0,1)}{-2\pi i\Omega_{\mathcal{G}}^{\mathrm{sign}(\eta\psi_0)}}\pmod{p^r}. \tag{8}$$

Note that if $\psi=\eta=\mathbf{1}$, by Lemma 4.3 the most we can say is

$$\Big|R(\mathcal{F},\mathbf{1})\;-\;R(\mathcal{G},\mathbf{1})\Big|_p\;\le\;\max\big\{p^{2w-r},\;p^{w-c}\big\} \tag{9}$$

where $\big|\alpha_1(p)\big|_p=\big|\alpha_2(p)\big|_p=p^{-w}$ and $\big|\alpha_1(p)^{-1}-\alpha_2(p)^{-1}\big|_p\le p^{-c}$. Equations (8) and (9) imply

$$\Big|\alpha_1(p)^{f(\rho,\mathfrak{p})}\cdot R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_1,\rho)\;-\;\alpha_2(p)^{f(\rho,\mathfrak{p})}\cdot R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_2,\rho)\Big|_p\;\le\;\begin{cases}\max\big\{p^{2w-r},\;p^{w-c}\big\} & \text{if } \psi=\mathbf{1}\\ p^{-r} & \text{if } \psi\neq\mathbf{1}.\end{cases} \tag{10}$$

Case I – $p\nmid t_p(E_\star)$: Since $t_p(E_1)\equiv t_p(E_2)\pmod{p^r}$ and we can write $t_p(E_\star)=\alpha_\star(p)+\frac{p}{\alpha_\star(p)}$ for $\star\in\{1,2\}$, it follows that

$$p^r\quad\text{divides}\quad t_p(E_1)-t_p(E_2)\;=\;\Big(\alpha_1(p)-\alpha_2(p)\Big)\times\left(1-\frac{p}{\alpha_1(p)\cdot\alpha_2(p)}\right).$$

As the right-hand bracket is a p-adic unit, thus $\Big|\alpha_1(p)-\alpha_2(p)\Big|_p=\Big|\alpha_1(p)^{-1}-\alpha_2(p)^{-1}\Big|_p\le p^{-r}$. One therefore has $w=0$, $c=r$ and $\alpha_1(p)^{f(\rho,\mathfrak{p})}\equiv\alpha_2(p)^{f(\rho,\mathfrak{p})}\pmod{p^r}$, in which case (10) becomes

$$\Big|R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_1,\rho)\;-\;R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_2,\rho)\Big|_p\;\le\;p^{-r}.$$

Case II – $p \mid t_p(E_\star)$ **and** $\psi \neq \mathbf{1}$**:** If $t_p(E_1) = t_p(E_2) = 0$, we can choose $\alpha_1(p) = \alpha_2(p) = \pm\sqrt{-p}$ in which case $w = 1/2$, $c = \infty$ and $\alpha_1(p)^{f(\rho,\mathfrak{p})} = \alpha_2(p)^{f(\rho,\mathfrak{p})}$, so that (10) becomes

$$\left| R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_1,\rho) \ - \ R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_2,\rho) \right|_p \ \leq \ p^{f(\rho,\mathfrak{p})\cdot \mathrm{ord}_p(\alpha_\star(p))} \times p^{-r} \ = \ p^{f(\rho,\mathfrak{p})/2-r}.$$

If $p = 3$ and $t_3(E_\star) \neq 0$, the upper bound must be weakened to $p^{f(\rho,\mathfrak{p})/2}$.

Case III – $p \mid t_p(E_\star)$ **and** $\psi = \mathbf{1}$**:** If $t_p(E_1) = t_p(E_2) = 0$, then the same choices as in the previous case imply the inequality (10) reduces to

$$\left| R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_1,\rho) \ - \ R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_2,\rho) \right|_p \ \leq \ p^{f(\rho,\mathfrak{p})/2} \times p^{\max\{2w-r,w-\infty\}} \ = \ p^{f(\rho,\mathfrak{p})/2+1-r}.$$

Unfortunately if $p = 3$ and $t_3(E_\star) \neq 0$, then the best we can do is

$$\left| R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_1,\rho) \ - \ R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_2,\rho) \right|_p \ < \ p^{f(\rho,\mathfrak{p})/2+1}$$

and in this situation, $\left| R_{S_1\cup S_2\cup\{\mathfrak{p}\}}(E_\star,\rho) \right|_p \leq p^{f(\rho,\mathfrak{p})/2+1}$ for each $\star \in \{1,2\}$.

Throughout we have been working with the set $S_1 \cup S_2 \cup \{\mathfrak{p}\}$. However if l is a rational prime not lying below this set, by assumption $t_l(E_1) \equiv t_l(E_2) \pmod{p^r}$, so the local L-factors satisfy

$$L_v(E_1,\rho,1) \ \equiv \ L_v(E_2,\rho,1) \pmod{p^r} \quad \text{at all places } v\big| l.$$

It follows that we may replace $S_1 \cup S_2 \cup \{\mathfrak{p}\}$ above with a larger set S containing it as a subset, yet still ensure that our congruences involving $R_S(E_\star,\rho)$ hold true.

Recall that the G_{F_n}-representation $\sigma_{F_n} \otimes \psi = \left(\mathrm{Ind}_{K_n}^{F_n}(\mathbf{1})\right) \otimes \psi = \psi \oplus \psi \cdot \varepsilon_{K_n/F_n}$ is reducible, and we have been assuming $t_m(E_1) \equiv t_m(E_2) \pmod{p^r}$ for all integers m coprime to N_1N_2. To summarise our arguments so far, we have established the following

Theorem 4.5. *Assume Hypothesis (Vat) holds for both f_1 and f_2, and that $S_1 \cup S_2 \cup \{\mathfrak{p}\} \subset S$.*

(i) If E_1 and E_2 have good ordinary reduction at p, then

$$R_S(E_1,\sigma_{F_n}\otimes\psi) \ \equiv \ R_S(E_1,\sigma_{F_n}\otimes\psi) \pmod{p^r}$$

and both sides of the congruence are p-integral.

(ii) If E_1 and E_2 have good supersingular reduction at p with $t_p(E_1) = t_p(E_2) = 0$, then

$$\left| R_S(E_1,\sigma_{F_n}\otimes\psi) \ - \ R_S(E_2,\sigma_{F_n}\otimes\psi) \right|_p \ \leq \ \begin{cases} p^{f(\rho,\mathfrak{p})/2+1-r} & \text{if } \psi = \mathbf{1} \\ p^{f(\rho,\mathfrak{p})/2-r} & \text{if } \psi \neq \mathbf{1} \end{cases}$$

where $\alpha_1(p) = \alpha_2(p) = \pm\sqrt{-p}$, and $\left| R_S(E_\star,\sigma_{F_n}\otimes\psi) \right|_p \leq \begin{cases} p^{f(\rho,\mathfrak{p})/2+1} & \text{if } \psi = \mathbf{1} \\ p^{f(\rho,\mathfrak{p})/2} & \text{if } \psi \neq \mathbf{1}. \end{cases}$

(iii) If E_1 and E_2 have good supersingular reduction at p with either $t_p(E_1) \neq 0$ or $t_p(E_2) \neq 0$,

$$\left| R_S(E_1,\sigma_{F_n}\otimes\psi) \ - \ R_S(E_2,\sigma_{F_n}\otimes\psi) \right|_p \ < \ p^{f(\rho,\mathfrak{p})/2+1}$$

provided that $\left|\alpha_1(p) - \alpha_2(p)\right|_p < p^{-1/2}$; here $\left| R_S(E_\star,\sigma_{F_n}\otimes\psi) \right|_p \leq \ p^{f(\rho,\mathfrak{p})/2+1}$ for $\star \in \{1,2\}$.

Note that (i) and (ii) essentially yield congruences mod p^r, whereas (iii) is a mod p congruence. It is unlikely that the latter can be improved much, unless the two roots $\alpha_1(p)$ and $\alpha_2(p)$ are sufficiently close to each other in $\mathcal{O}_{\mathbb{C}_p}$.

Indeed the congruence in (ii) suggests that any (hypothetical!) pair of p-adic L-functions attached to E_1 and E_2 over K_n should satisfy a congruence mod p^r whenever f_1 and f_2 do, even though such p-bounded measures do not at present exist.

5 Combining together three separate congruences

To complete the proof of Theorems 1.2 and 1.3, it is enough to establish mod p congruences linking the following three pairs:

(i) $R_S(E_1, \rho_{F_n})$ and $R_S(E_1, \sigma_{F_n})$;

(ii) $R_S(E_1, \sigma_{F_n})$ and $R_S(E_2, \sigma_{F_n})$;

(iii) $R_S(E_2, \sigma_{F_n})$ and $R_S(E_2, \rho_{F_n})$.

We first need to check under what conditions both $R_S(E_1, \rho_{F_n})$ and $R_S(E_2, \rho_{F_n})$ are p-integral in the case of ordinary reduction, and have controlled growth in the supersingular case. Given the period issues for the σ_{F_n}-twists are resolved through Vatsal's choices of period in the previous section, morally the same periods should work nicely for the ρ_{F_n}-twists.

5.1 Modular symbols over CM fields

Let E be a semistable elliptic curve over $\mathbb{Q}$. We begin by rewriting the Hasse-Weil L-values for E in terms of modular symbols over the CM-extension $K_n = \mathbb{Q}(\mu_{p^n})$ of F_n (see [Har87, Bou06]). Consider the algebraic group $\mathbb{G} = \mathrm{Res}_{K_n/\mathbb{Q}}\mathrm{GL}(2)$ given by restricting scalars from K_n to $\mathbb{Q}$.

Let $\pi_{E,n}$ denote the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{K_n})$, corresponding to the base-change of the associated newform $f_E \in \mathcal{S}_2\big(\Gamma_0(N_E)\big)$ up to K_n. Then there exists an adelic L^2-function $\mathfrak{F} = \mathfrak{F}_{/K_n} : \mathbb{G}(\mathbb{A}) \to V_{\mathbb{C}}$, which is the projection of the canonical primitive form associated to $\pi_{E,n} = \bigotimes_\nu \pi_{E,n}^{(\nu)}$; here the complex vector space $V_{\mathbb{C}}$ is $3^{[K_n:\mathbb{Q}]}$-dimensional.

If $\mathfrak{F}_0 = \mathfrak{F}\big|_\infty$ corresponds to the archimedean part $\bigotimes_{\nu|\infty} \pi_{E,n}^{(\nu)}$, by choosing sub-coordinates (x, y) appropriately, we can view this as an L^2-function $\mathfrak{F}_0 : \mathbb{A}_{K_n}^\times \times \big(K_n \otimes \hat{\mathbb{Z}}\big) \to \mathbb{C}$.

Definition 5.1. *For each $\mathfrak{r} \in \mathbb{A}_{K_n}^\times$ and $\vartheta \in K_n \otimes \big(\varprojlim_n \mathbb{Z}/n\mathbb{Z}\big)$, one associates the symbol*

$$\mathcal{MS}_E(\mathfrak{r}, \vartheta) \;=\; \frac{1}{\big[\mathcal{O}_{K_n}^\times : \mathcal{E}_n\big]} \times \int_{x \in \mathfrak{X}_{n,\mathcal{E}}} \mathfrak{F}_0\big(\mathfrak{r}\mathfrak{d}_{K_n} x, -\vartheta\big) \cdot \mathrm{d}^\times x$$

where $\mathfrak{X}_{n,\mathcal{E}} := K_{n,\infty\,+}^\times \times \prod_\nu \mathcal{O}_{K_{n,\nu}} \big/ \mathcal{E}_n$, and $\mathcal{E}_n = \mathcal{E}_n(\mathfrak{r}, \vartheta)$ is any subgroup of totally positive units $\mathfrak{u}$ of finite index inside the ring $\mathcal{O}_{F_n}$, such that $(1 - \mathfrak{u})\vartheta \in \mathfrak{r} \cdot \prod_\nu \mathcal{O}_{K_{n,\nu}}$.

The above integrals are certainly well-defined, and independent of the choice of subgroup $\mathcal{E}_n$. Also the Haar measure $\mathrm{d}^\times x = \otimes_\nu \mathrm{d}^\times x_\nu$ has itself been normalised so that $\int_{\mathcal{O}_{K_{n,\nu}}} \mathrm{d}^\times x_\nu = 1$ at the finite places, whilst $\mathrm{d}^\times x_\nu = \frac{\mathrm{d}\,\mathrm{sgn}(x_\nu) \wedge \mathrm{d}|x|_\nu}{2\pi i \cdot x_\nu}$ at every archimedean place.

Proposition 5.2. *(Birch's Lemma) If $\tilde{\chi}$ is a Hecke character over K_n of conductor $\mathfrak{c}_{\tilde{\chi}}$, then*

$$L(\pi_E, \tilde{\chi}, 1) \;=\; (2\pi)^{[K_n:\mathbb{Q}]} \tau_{K_n}(\tilde{\chi}) \cdot N_{K_n/\mathbb{Q}}(\mathfrak{c}_{\tilde{\chi}})^{-1} \;\times \sum_{j=1}^{\#\mathrm{Cl}(\mathcal{O}_{K_n})} \;\sum_{\vartheta \in (\mathcal{O}_{K_n}/\mathfrak{c}_{\tilde{\chi}})^\times} \tilde{\chi}(\vartheta \mathfrak{r}_j) \mathcal{MS}_E(\mathfrak{r}_j \mathfrak{c}_{\tilde{\chi}}, \vartheta)$$

where the $\mathfrak{r}_j$'s run over a set of representatives for $\mathrm{Cl}(\mathcal{O}_{K_n})$.

Proof. This is demonstrated in [Har87, Section 4] over an arbitrary CM field. □

In particular, if $\psi : \Gamma_n^{\mathrm{cy}} \to \mathbb{C}^\times$ has finite order and if $\tilde{\chi} = \begin{cases} \chi \otimes \mathrm{Res}_{K_n}(\psi) & \text{if } \varrho = \rho_{F_n} \\ \mathbf{1}_{K_n, \mathfrak{l}_0} \otimes \mathrm{Res}_{K_n}(\psi) & \text{if } \varrho = \sigma_{F_n,S}, \end{cases}$ then $L(\pi_E, \tilde{\chi}, s) = L_{S_0}(E, \varrho \otimes \psi^{-1}, s)$. Now one can identify $\frac{N_{K_n/\mathbb{Q}}(\mathfrak{c}_{\tilde{\chi}})}{\tau_{K_n}(\tilde{\chi})}$ with $\mathrm{sgn}(\tilde{\chi}) \cdot \tau_{K_n}\big(\tilde{\chi}^{-1}\big)$

and then by inductivity, the Gauss sum $\tau_{K_n}\left(\tilde{\chi}^{-1}\right)$ coincides with the epsilon factor $\epsilon_{F_n}\left(\varrho\otimes\psi^{-1}\right)$. Dividing through by the period, Birch's Lemma can therefore be rewritten as

$$\sum_{j=1}^{\#\mathrm{Cl}(\mathcal{O}_{K_n})} \sum_{\vartheta\in(\mathcal{O}_{K_n}/\mathfrak{c}_{\tilde{\chi}})^\times} \tilde{\chi}(\vartheta\mathfrak{r}_j)\cdot\frac{(2\pi)^{[K_n:\mathbb{Q}]}\mathcal{MS}_E(\mathfrak{r}_j\mathfrak{c}_{\tilde{\chi}},\vartheta)}{(-4\pi^2\Omega^+_{f_E}\Omega^-_{f_E})^{[F_n:\mathbb{Q}]}} = \mathrm{sgn}(\tilde{\chi})\cdot\epsilon_{F_n}\left(\varrho\otimes\psi^{-1}\right)\times\frac{L_{S_0}(E,\varrho\otimes\psi^{-1},1)}{(-4\pi^2\Omega^+_{f_E}\Omega^-_{f_E})^{[F_n:\mathbb{Q}]}}. \tag{11}$$

Remarks: (i) A result of Haran [Har87, p37] states that the submodule $\mathcal{L}^0$ of $\mathbb{C}$ generated by $[\mathcal{O}^\times_{K_n}:\mathcal{E}_n]\cdot\mathcal{MS}_E(\mathfrak{r},\vartheta)$ with $|\vartheta|_v<|\mathfrak{r}_j|_v$, is of finite-type over $\mathbb{Z}$; if $\left\{(\mathfrak{r}_1,\vartheta_1),\ldots,(\mathfrak{r}_t,\vartheta_t)\right\}$ yields a full set of generators for $\mathcal{L}^0$, one can then define the p-power number

$$\mathfrak{I}_{\mathcal{E}_n,p}(K_n) \;:=\; \sup\left\{\Big|\big[\mathcal{O}^\times_{K_n}:\mathcal{E}_n(\mathfrak{r}_i,\vartheta_i)\big]\Big|_p^{-1} \text{ with } i=1,\ldots,t\right\}.$$

(ii) In Bouganis' thesis [Bou06, p84] it is shown that if $\xi_{\mathfrak{F}}$ generates the free $\mathbb{Z}_{(p)}$-module $\mathrm{Im}\Big(H^{[K_n:\mathbb{Q}]}_{\mathrm{cusp}}\big(Y_0(N_E),\mathbb{Z}_{(p)}\big)\;\to\; H^{[K_n:\mathbb{Q}]}_{\mathrm{cusp}}\big(Y_0(N_E),\mathbb{Q}\big)\Big)\big[\mathfrak{F}\big]$, there exists an automorphic period $\Omega^{\mathrm{aut}}_{E,p}(\mathfrak{F})$ such that $\xi_{\mathfrak{F}}=\Omega^{\mathrm{aut}}_{E,p}(\mathfrak{F})\cdot d\omega_{\mathfrak{F}}$, where $d\omega_{\mathfrak{F}}$ is the differential corresponding to $\mathfrak{F}$.

(iii) For a fixed commutative ring R, the inverse of $\tau^{\mathrm{MD}}_{R,p}=\tau^{\mathrm{MD}}_{R,p}(K_n)\in R_{(p)}$ refers to the denominator occurring in the section

$$1-\mathrm{Eis}: H^d_{\mathrm{cusp}}\big(Y_0(N_E),R_{(p)}\big)\longrightarrow H^d_{\mathrm{c}}\big(Y_0(N_E),R_{(p)}\big) \quad\text{with } d=[K_n:\mathbb{Q}];$$

here 'Eis' above denotes the Hecke idempotent which cuts out the Eisenstein portion of the Manin-Drinfeld splitting in degree $2d$.

(iv) Lastly, it is then explained in [Bou06, §4.4] that there is a natural containment

$$\iota_p\left(\Omega^{\mathrm{aut}}_{E,p}(\mathfrak{F})^{-1}\cdot\mathcal{L}^0\right)\otimes_{\mathbb{Z}}\mathbb{Z}_p \;\subset\; \tau^{\mathrm{MD}}_{\mathbb{Z},p}\cdot\mathcal{O}_{\mathbb{C}_p}.$$

Definition 5.3. *For every* $\mathfrak{r}\in\mathbb{A}^\times_{K_n}$ *and* $\vartheta\in K_n\otimes\Big(\varprojlim_n\mathbb{Z}/n\mathbb{Z}\Big)$ *as before, let us introduce the modified symbol*

$$\mathcal{MS}^{\mathrm{vat}}_E(\mathfrak{r},\vartheta) \;:=\; \frac{(2\pi)^{[K_n:\mathbb{Q}]}}{(-4\pi^2\Omega^+_{f_E}\Omega^-_{f_E})^{[F_n:\mathbb{Q}]}}\times\mathcal{MS}_E(\mathfrak{r},\vartheta).$$

The advantage of making this modification is that $\mathcal{MS}^{\mathrm{vat}}_E(-,-)$ will take algebraic values, rather than purely transcendental ones as per $\mathcal{MS}_E(-,-)$.

Corollary 5.4. *At each* ψ *of* $\mathfrak{p}$*-power conductor with* $\tilde{\chi}=\begin{cases}\chi\otimes\mathrm{Res}_{K_n}(\psi) & \text{if } \varrho=\rho_{F_n}\\ \mathbf{1}_{K_n,\mathfrak{l}_0}\otimes\mathrm{Res}_{K_n}(\psi) & \text{if } \varrho=\sigma_{F_n,S},\end{cases}$

$$\sum_{j=1}^{\#\mathrm{Cl}(\mathcal{O}_{K_n})}\sum_{\vartheta\in(\mathcal{O}_{K_n}/\mathfrak{c}_{\tilde{\chi}})^\times}\tilde{\chi}(\vartheta\mathfrak{r}_j)\times\mathcal{MS}^{\mathrm{vat}}_E(\mathfrak{r}_j\mathfrak{c}_{\tilde{\chi}},\vartheta) \;\in\; \iota_\infty^{-1}\left(\frac{\tau^{\mathrm{MD}}_{\mathbb{Z},p}(K_n)}{\mathfrak{I}_{\mathcal{E}_n,p}(K_n)}\times\frac{\Omega^{\mathrm{aut}}_{E,p}(\mathfrak{F})}{\left(\Omega^+_{f_E}\Omega^-_{f_E}\right)^{[F_n:\mathbb{Q}]}}\right)\cdot\mathcal{O}_{\mathbb{C}_p}.$$

Proof. One simply combines together Remarks (i)-(iv) above. □

We now return to the main situation in the paper, so again E_1 and E_2 are two semistable elliptic curves whose Fourier coefficients satisfy $t_m(E_1)\equiv t_m(E_2) \pmod p$ if $\gcd(m,N_1N_2)=1$.

Hypothesis (MS)$_p$. *For both choices $\star\in\{1,2\}$, the $\mathbb{Z}_p$-lattice generated by the values*

$$\sum_{j=1}^{\#\mathrm{Cl}(\mathcal{O}_{K_n})}\ \sum_{\vartheta\in(\mathcal{O}_{K_n}/\mathfrak{c}_{\tilde\chi})^\times}\tilde\chi(\vartheta\mathfrak{r}_j)\times\mathcal{MS}^{\mathrm{vat}}_{E_\star}(\mathfrak{r}_j\mathfrak{c}_{\tilde\chi},\vartheta)$$

where $\tilde\chi$ ranges over all Hecke characters as above, is properly contained inside $\mathcal{O}_{\mathbb{C}_p}$.

We remark that this hypothesis ensures that the algebraic L-values we study are p-integral. Should these values turn out to be non-integral (which is clearly **not** suggested by the numerical data compiled in [DD07] and in §3.3), then it is possible to modify our arguments to show that analogous congruence relations hold, after rescaling the p-adic L-values appropriately – see §5.5 below for a further discussion.

Corollary 5.4 supplies us with, of course, a larger lattice that these values are contained in. Moreover if one assumes that Hypothesis (Vat) is true, using Proposition 4.1 one can show

$$\sum_{j=1}^{\#\mathrm{Cl}(\mathcal{O}_{K_n})}\ \sum_{\vartheta\in(\mathcal{O}_{K_n}/\mathfrak{c}_{\tilde\chi})^\times}\tilde\chi(\vartheta\mathfrak{r}_j)\times\mathcal{MS}^{\mathrm{vat}}_{E_\star}(\mathfrak{r}_j\mathfrak{c}_{\tilde\chi},\vartheta)\ \in\ \mathcal{O}_{\mathbb{C}_p}$$

for all characters $\tilde\chi$ of the form $\mathbf{1}_{K_n,\mathfrak{l}_0}\otimes\mathrm{Res}_{K_n}(\psi)$ – we leave this as an exercise for the reader. Henceforth we assume our finite set S of places is chosen large enough, so that

$$S_0\cup S_1\cup S_2\ \subset\ S.$$

Recall in our calculations with the functional $\mathcal{L}^\alpha_{\star,F_n}(-)$, we made a minimal choice of exponents

$$\mathfrak{m}'=\mathfrak{p}^{f(\rho_{F_n}\otimes\psi,\mathfrak{p})}\quad\text{and}\quad\mathfrak{l}'=\mathfrak{c}(\mathbf{g}_{\rho_{F_n}\otimes\psi})\cdot\mathfrak{p}^{-f(\rho_{F_n}\otimes\psi,\mathfrak{p})}\cdot\mathfrak{l}_0^2\,,$$

and set $\mathcal{T}_{\star,F_n}:=\max\left\{\mathcal{T}^{(\mathfrak{m}'\mathfrak{l}')}_{\star,F_n}(\rho_{F_n}),\mathcal{T}^{(\mathfrak{m}'\mathfrak{l}')}_{\star,F_n}(\sigma_{F_n,S})\right\}$ where $\mathcal{T}^{(\mathfrak{m}'\mathfrak{l}')}_{\star,F_n}(\varrho)$ is defined in Equation (3). The following result was used to obtain both Equation (4) and (3.2.1)-(3.2.3) in Section 3.2.

Proposition 5.5. *If Hypothesis (MS)$_p$ holds true, then $\mathcal{T}_{\star,F_n}\le 1$ for both $\star\in\{1,2\}$.*

Proof. If $\rho=\rho_{F_n}\otimes\psi$ or $\rho=\sigma_{F_n,S}\otimes\psi$ with ψ factoring through Γ^{cy}_n, then

$$\mathbf{K}_{\star,F_n}\times\mathcal{L}^\alpha_{\star,F_n}\left(\Phi^{(n)}_\rho\Big|U\big(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1}\big)\right)\overset{\text{by 3.6}}{=}\alpha_\star(\mathfrak{m}'\mathfrak{l}')\times R_S\big(E_\star\,,\rho\big)$$

$$\overset{\text{by 1.1}}{=}\frac{\alpha_\star(\mathfrak{m}'\mathfrak{l}')}{\alpha_\star(\mathfrak{p})^{f(\rho,\mathfrak{p})}}\cdot\left(\frac{P_\mathfrak{p}\big(\rho,\alpha_\star(p)^{-1}\big)}{P_\mathfrak{p}\big(\rho^*,\beta_\star(p)^{-1}\big)}\cdot\prod_{v\in S\backslash S_0}L_v(E_\star\,,\rho^*,1)\right)\times\frac{\epsilon_{F_n}(\rho^*)\ L_{S_0}(E_\star\,,\varrho^*,1)}{(-4\pi^2\Omega^+_{f_E}\Omega^-_{f_E})^{[F_n:\mathbb{Q}]}}\,.$$

Clearly $\mathrm{ord}_p\left(\frac{\alpha_\star(\mathfrak{m}'\mathfrak{l}')}{\alpha_\star(\mathfrak{p})^{f(\rho,\mathfrak{p})}}\right)\ge 0$ by our choice of $\mathfrak{m}'$, while the bracketted term is also p-integral. Finally, using the variant of Birch's Lemma in Equation (11), the right-most term becomes

$$\frac{\epsilon_{F_n}(\rho^*)\ L_{S_0}(E_\star\,,\varrho^*,1)}{(-4\pi^2\Omega^+_{f_E}\Omega^-_{f_E})^{[F_n:\mathbb{Q}]}}\ =\ \mathrm{sgn}(\tilde\chi)\cdot\sum_{j=1}^{\#\mathrm{Cl}(\mathcal{O}_{K_n})}\ \sum_{\vartheta\in(\mathcal{O}_{K_n}/\mathfrak{c}_{\tilde\chi})^\times}\tilde\chi(\vartheta\mathfrak{r}_j)\times\mathcal{MS}^{\mathrm{vat}}_{E_\star}(\mathfrak{r}_j\mathfrak{c}_{\tilde\chi},\vartheta)$$

which is p-integral under Hypothesis (MS)$_p$. It follows immediately that

$$\left|\mathbf{K}_{\star,F_n}\times\mathcal{L}^\alpha_{\star,F_n}\left(\Phi^{(n)}_\rho\Big|U\big(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1}\big)\right)\right|_p\ \le\ \left|\frac{\alpha_\star(\mathfrak{m}'\mathfrak{l}')}{\alpha_\star(\mathfrak{p})^{f(\rho,\mathfrak{p})}}\right|_p\ \le\ 1$$

for both $\rho=\rho_{F_n}\otimes\psi$ or $\rho=\sigma_{F_n,S}\otimes\psi$; in other words $\mathcal{T}_{\star,F_n}\le 1$, as required. □

5.2 An improvement in the p-ordinary case

The following mod p congruence relies on the crucial observation that in the ordinary case, there exist p-bounded measures interpolating the special values of $h^1(E_\star)\otimes\rho_{F_n}$ and $h^1(E_\star)\otimes\sigma_{F_n}$ over the character-space for the cyclotomic $\mathbb{Z}_p$-extension of K_n.

Unfortunately in the case of supersingular reduction, no one has yet constructed bounded p-adic L-functions interpolating these L-values, so we cannot make any improvements here. David Loeffler and Mahesh Kakde pointed out that whilst the p-adic distributions in [Har87] are $1/2$-admissible on $\big(\mathcal{O}_{F_n}\otimes\mathbb{Z}_p\big)^\times$, once one projects to the cyclotomic component $\mathrm{Gal}(F_\infty/F_n)$ the corresponding measure is only $\big([F_n:\mathbb{Q}]/2\big)$-admissible on Γ_n^{cy}, hence not uniquely defined.

Theorem 5.6. *(i) If $E_\star$ has good ordinary reduction at p and Hypothesis (MS)$_p$ holds, then*

$$R_S\big(E_\star\,,\rho_{F_n}\otimes\psi\big)\;\equiv\;R_S\big(E_\star\,,\sigma_{F_n}\otimes\psi\big)\quad\mathrm{mod}\;p$$

at all finite characters $\psi:\mathrm{Gal}(K_\infty/F_n)\to\overline{\mathbb{Q}}_p^\times$, and both sides of the congruence are p-integral.

(ii) Furthermore, if Hypothesis (Vat) holds too then at infinitely many such characters ψ above, this mod p congruence is non-trivial.

Proof. There exist p-adic L-functions $\mathbf{L}_p\big(E_\star\,,\rho_{F_n}\big)$ and $\mathbf{L}_p\big(E_\star\,,\sigma_{F_n}\big)$ in $\mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/K_n)]\!]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ satisfying at all such ψ above,

$$\begin{aligned}\psi\left(\mathbf{L}_p(E_\star\,,\rho_{F_n})\right)\;&=\;\iota_p\Big(R_S\big(E_\star\,,\rho_{F_n}\otimes\psi\big)\Big)\\ \text{and}\qquad\psi\left(\mathbf{L}_p(E_\star\,,\sigma_{F_n})\right)\;&=\;\iota_p\Big(R_S\big(E_\star\,,\sigma_{F_n}\otimes\psi\big)\Big).\end{aligned}$$

This was proven in [Bou06, DW08], albeit with a different choice of periods from Vatsal's ones, and for the (possibly) smaller choice of set $S_0\subset S$.

Each value $R_S\big(E_\star\,,\sigma_{F_n}\otimes\psi\big)$ is then p-integral because we are assuming Hypothesis (MS)$_p$, and these can vanish for only finitely many ψ since $\mathbf{L}_p\big(E_\star\,,\sigma_{F_n}\big)\neq 0$. The latter statement follows from a well known result of Rohrlich [Roh88], implying the non-vanishing of $L(E_\star\,,\psi,1)$ at all but finitely many Dirichlet twists ψ factoring through the extension $K_\infty/\mathbb{Q}$.

By Equation (4), for all such characters ψ we deduce that $R_S\big(E_\star\,,\rho_{F_n}\otimes\psi\big)$ is also p-integral. In fact under Hypothesis (Vat), the values $R_S\big(E_\star\,,\sigma_{F_n}\otimes\psi\big)$ are not divisible by p for infinitely many characters ψ, so the same must hold true for the collection of L-values $R_S\big(E_\star\,,\rho_{F_n}\otimes\psi\big)$. It follows that $\mathbf{L}_p\big(E_\star\,,\rho_{F_n}\big)$ is non-zero, and moreover

$$\mathbf{L}_p\big(E_\star\,,\rho_{F_n}\big)-\mathbf{L}_p\big(E_\star\,,\sigma_{F_n}\big)\in\;\mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/K_n)]\!]\cap\mathfrak{M}_{\mathbb{C}_p}[\![\mathrm{Gal}(K_\infty/K_n)]\!]\;=\;p\cdot\mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/K_n)]\!].$$

As a direct consequence, $\psi\big(\mathbf{L}_p\big(E_\star\,,\rho_{F_n}\big)\big)-\psi\big(\mathbf{L}_p\big(E_\star\,,\sigma_{F_n}\big)\big)\in p\cdot\mathcal{O}_{\mathbb{C}_p}$ at all characters ψ. □

5.3 Proof of Theorems 1.2 and 1.3

The preparatory work is over, and we can now prove the two main results in the Introduction. Recall that the elliptic curves E_1 and E_2 have good reduction at the prime p, their Fourier coefficients satisfy $t_m(E_1)\equiv t_m(E_2)\pmod p$ if $\gcd(m,N_1N_2)=1$, and also $S_0\cup S_1\cup S_2\subset S$. **Throughout we suppose that Hypotheses (Vat) and (MS)$_p$ are true.**

Firstly, the integrality statements in Theorems 1.2(i) and 1.3(i) follow from Proposition 5.5. If we abbreviate $\rho_{F_n}\otimes\psi$ by $\rho_{n,\psi}$, and $\sigma_{F_n}\otimes\psi$ by $\sigma_{n,\psi}$, then $\big|R_S(E_1,\rho_{n,\psi})-R_S(E_2,\rho_{n,\psi})\big|_p$ will be equal to

$$\Big|R_S(E_1,\rho_{n,\psi}) - R_S(E_1,\sigma_{n,\psi}) + R_S(E_1,\sigma_{n,\psi}) - R_S(E_2,\sigma_{n,\psi}) + R_S(E_2,\sigma_{n,\psi}) - R_S(E_2,\rho_{n,\psi})\Big|_p$$
$$\leq \max\Big\{\big|R_S(E_1,\sigma_{n,\psi}) - R_S(E_2,\sigma_{n,\psi})\big|_p\,,\, \big|R_S(E_\star\,,\rho_{n,\psi}) - R_S(E_\star\,,\sigma_{n,\psi})\big|_p \text{ for each } \star\in\{1,2\}\Big\}$$

upon applying the strong triangle inequality. Therefore, we just need to find upper bounds for

$$\big|R_S(E_1,\sigma_{n,\psi}) - R_S(E_2,\sigma_{n,\psi})\big|_p \quad\text{and}\quad \big|R_S(E_\star\,,\rho_{n,\psi}) - R_S(E_\star\,,\sigma_{n,\psi})\big|_p\,,$$

which we shall do on a case-by-case basis.

Case 1 – $p\nmid t_p(E_\star)$: Since E_1 and E_2 have ordinary reduction at p, by Theorem 4.5(i) one has

$$\big|R_S(E_1,\sigma_{n,\psi}) - R_S(E_2,\sigma_{n,\psi})\big|_p \;\leq\; p^{-1},$$

while Theorem 5.6(i) implies

$$\big|R_S(E_\star\,,\rho_{n,\psi}) - R_S(E_\star\,,\sigma_{n,\psi})\big|_p \;\leq\; p^{-1}$$

hence Theorem 1.2(ii) now follows. The non-triviality assertion for infinitely many ψ-twists is itself an immediate consequence of Theorem 5.6(ii).

Case 2a – $p\mid t_p(E_\star)$, $t_p(E_1)=t_p(E_2)=0$ and $\psi\neq\mathbf{1}$: Using Theorem 4.5(ii), we deduce that

$$\big|R_S(E_1,\sigma_{n,\psi}) - R_S(E_2,\sigma_{n,\psi})\big|_p \;\leq\; p^{f(\sigma_{n,\psi},\mathfrak{p})/2-1}.$$

Case 2b – $p\mid t_p(E_\star)$, $t_p(E_1)=t_p(E_2)=0$ and $\psi=\mathbf{1}$: Again by Theorem 4.5(ii), one finds

$$\big|R_S(E_1,\sigma_{n,\psi}) - R_S(E_2,\sigma_{n,\psi})\big|_p \;\leq\; p^{f(\sigma_{n,\psi},\mathfrak{p})/2}.$$

Case 3 – $p\mid t_p(E_\star)$, $t_p(E_1)\neq 0$ or $t_p(E_2)\neq 0$: From Theorem 4.5(iii), we conclude that

$$\big|R_S(E_1,\sigma_{n,\psi}) - R_S(E_2,\sigma_{n,\psi})\big|_p \;<\; p^{f(\sigma_{n,\psi},\mathfrak{p})/2+1}.$$

Note in each of Cases 2a, 2b and 3, one additionally knows from Theorem 3.7 that

$$\big|R_S(E_\star\,,\rho_{n,\psi}) - R_S(E_\star\,,\sigma_{n,\psi})\big|_p \;<\; p^{f(\rho_{n,\psi},\mathfrak{p})/2}.$$

To complete the proof of Theorem 1.3(ii), one simply observes that $f(\sigma_{n,\psi},\mathfrak{p})+1<f(\rho_{n,\psi},\mathfrak{p})$, unless one has $n=1$, $\psi=\mathbf{1}$, $\Delta^{p-1}\equiv 1 \pmod{p^2}$ in which case $f(\sigma_{n,\mathbf{1}},\mathfrak{p})=f(\rho_{n,\mathbf{1}},\mathfrak{p})=p-2$ (see Lemma 3.8 for the details).

Putting everything together in the supersingular case, if $p\mid t_p(E_\star)$ then

$$\big|R_S(E_1,\rho_{n,\psi}) - R_S(E_2,\rho_{n,\psi})\big|_p \;<\; \max\Big\{p^{f(\sigma_{n,\psi},\mathfrak{p})/2+1},\, p^{f(\rho_{n,\psi},\mathfrak{p})/2}\Big\} \;=\; p^{f(\rho_{n,\psi},\mathfrak{p})/2}.$$

The demonstration of the two theorems from the Introduction is now finished.

5.4 An application to the μ- and λ-invariants

As already mentioned in the proof of Theorem 5.6, when p is an ordinary prime for E_1 and E_2, the values $R_S(E_\star,\rho_{F_n}\otimes\psi)$ are interpolated by the p-adic L-function $\mathbf{L}_p(E_\star,\rho_{F_n})$ as ψ varies. Using our modulo p congruence relation, one may deduce the following equalities between the Iwasawa invariants of $\mathbf{L}_p(E_1,\rho_{F_n})$ and $\mathbf{L}_p(E_2,\rho_{F_n})$.

Theorem 5.7. *If E_1 and E_2 have ordinary reduction at p, and Hypotheses (MS)$_p$ and (Vat) hold true, then*

(a) the μ-invariants of $\mathbf{L}_p\big(E_1, \rho_{F_n}\big)$ and $\mathbf{L}_p\big(E_2, \rho_{F_n}\big)$ are both zero;

(b) the λ-invariants of $\mathbf{L}_p\big(E_1, \rho_{F_n}\big)$ and $\mathbf{L}_p\big(E_2, \rho_{F_n}\big)$ will be equal.

Proof. We recall from the proof of Theorem 5.6 that Hypothesis (Vat) implies the algebraic integers $R_S(E_\star, \rho_{F_n} \otimes \psi)$ are not divisible by p for infinitely many characters ψ, which establishes assertion (a). Statement (b) itself is an immediate consequence of [DL15, Lemma 2.1]. □

We point out that this theorem is a direct analogue of Emerton, Pollack and Weston's results, concerning the Iwasawa invariants of p-adic L-functions for modular forms belonging to the same Hida family (c.f. [EPW06, Theorems 1 and 2]).

5.5 Remarks on our hypotheses

Note that the Iwasawa main conjecture predicts the ρ_{F_n}-twisted Selmer group for $E_\star$ over the cyclotomic $\mathbb{Z}_p$-extension should be a $\mathbb{Z}_p\big[\big[\Gamma_n^{\text{cy}}\big]\big]$-cotorsion module; furthermore, the characteristic power series of its Pontrjagin dual (conjecturally) equals $\mathbf{L}_p\big(E_\star, \rho_{F_n}\big)$, up to a unit of $\mathbb{Z}_p\big[\big[\Gamma_n^{\text{cy}}\big]\big]$. Therefore, if the ρ_{F_n}-twisted Iwasawa main conjecture holds true then Hypothesis (MS)$_p$ must automatically be true as well, since a characteristic power series is always p-integral.

One should, of course, ask what occurs if either of Hypotheses (Vat) or (MS)$_p$ do *not* hold – here one needs to modify Theorems 1.2 and 1.3, replacing $R_S(E_\star\,, \varrho)$ with the rescaled version

$$R_S^\dagger(E_\star\,, \varrho) \;:=\; \mathcal{T}_{\star,F_n} \times R_S(E_\star\,, \varrho).$$

The arguments outlined in this paper still work fine, and our results now become hypothesis-free.

Finally, there is no need to insist that f_1 and f_2 only satisfy a mod p congruence; instead one can assume their coefficients are such that $t_m(E_1) \equiv t_m(E_2) \pmod{p^r}$ if $\gcd(m, N_1N_2) = 1$, which is a stronger condition when $r > 1$ than before.

Conjecture 5.8. *If E_1 and E_2 have good reduction at $p > 2$, and if $S_0 \cup S_1 \cup S_2 \subset S$, then*

$$\Big| R_S(E_1, \rho_{F_n} \otimes \psi) - R_S(E_2, \rho_{F_n} \otimes \psi) \Big|_p \;\;\leq\;\; \begin{cases} p^{-r} & \text{if } p \nmid t_p(E_\star) \\ p^{-r+f(\rho_{F_n}\otimes\psi,\mathfrak{p})/2} & \text{if } p \mid t_p(E_\star). \end{cases}$$

To establish this prediction for each $r \geq 2$, would require a radically different approach to that outlined in this article. Nevertheless, it is certainly a problem ripe for numerical investigation.

References

[BDP13] Massimo Bertolini, Henri Darmon, and Kartik Prasanna, *Generalized Heegner cycles and p-adic Rankin L-series*, Duke Math. J. **162** (2013), no. 6, 1033–1148, With an appendix by Brian Conrad.

[Bou06] Thanasis Bouganis, *L-functions of elliptic curves and false Tate curve extensions*, Ph.D. thesis, University of Cambridge, 2006.

[CK16] Suh Hyun Choi and Byoung Du Kim, *Congruences of two-variable p-adic l-functions of congruent modular forms of different weights*, to appear in Ramanujan J, 2016.

[DD07] Tim Dokchitser and Vladimir Dokchitser, *Computations in non-commutative Iwasawa theory*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 1, 211–272, With an appendix by J. Coates and R. Sujatha.

[DL15] Daniel Delbourgo and Antonio Lei, *Transition formulae for ranks of abelian varieties*, Rocky Mountain J. Math. **45** (2015), no. 6, 1807–1838.

[DW08] Daniel Delbourgo and Thomas Ward, *Non-abelian congruences between L-values of elliptic curves*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 3, 1023–1055.

[DW10] ———, *The growth of CM periods over false Tate extensions*, Experiment. Math. **19** (2010), no. 2, 195–210.

[EPW06] Matthew Emerton, Robert Pollack, and Tom Weston, *Variation of Iwasawa invariants in Hida families*, Invent. Math. **163** (2006), no. 3, 523–580.

[GV00] Ralph Greenberg and Vinayak Vatsal, *On the Iwasawa invariants of elliptic curves*, Invent. Math. **142** (2000), no. 1, 17–63.

[Har87] Shai Haran, *p-adic L-functions for modular forms*, Compositio Math. **62** (1987), no. 1, 31–46.

[Kim14] Byoung Du Kim, *Two-variable p-adic L-functions of modular forms for non-ordinary primes*, J. Number Theory **144** (2014), 188–218.

[KL16] Daniel Kriz and Chao Li, *Congruences between Heegner points and quadratic twists of elliptic curves*, 2016, arXiv:1606.03172.

[Pan91] Alexey A. Panchishkin, *Non-Archimedean L-functions of Siegel and Hilbert modular forms*, Lecture Notes in Mathematics, vol. 1471, Springer-Verlag, Berlin, 1991.

[Roh88] David E. Rohrlich, *L-functions and division towers*, Math. Ann. **281** (1988), no. 4, 611–632.

[Ser87] Jean-Pierre Serre, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230.

[Shi78] Goro Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J. **45** (1978), no. 3, 637–679.

[SS14] Sudhanshu Shekhar and R. Sujatha, *Euler characteristic and congruences of elliptic curves*, Münster J. Math. **7** (2014), no. 1, 327–343.

[SS15] ———, *Congruence formula for certain dihedral twists*, Trans. Amer. Math. Soc. **367** (2015), no. 5, 3579–3598.

[Vat99] V. Vatsal, *Canonical periods and congruence formulae*, Duke Math. J. **98** (1999), no. 2, 397–419.

[War12] Thomas Ward, *Congruences for convolutions of Hilbert modular forms*, Math. Proc. Cambridge Philos. Soc. **153** (2012), no. 3, 471–487.

[Was97] Larry Washington, *Introduction to cyclotomic fields*, second ed., Graduate Texts in Mathematics, vol. 83, Springer Verlag, New York, 1997.

[Wes10] Jacob Westlund, *On the fundamental number of the algebraic number field $k(\sqrt[p]{m})$*, Trans. Am. Math. Soc. **11** (1910), no. 4, 388–392.

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